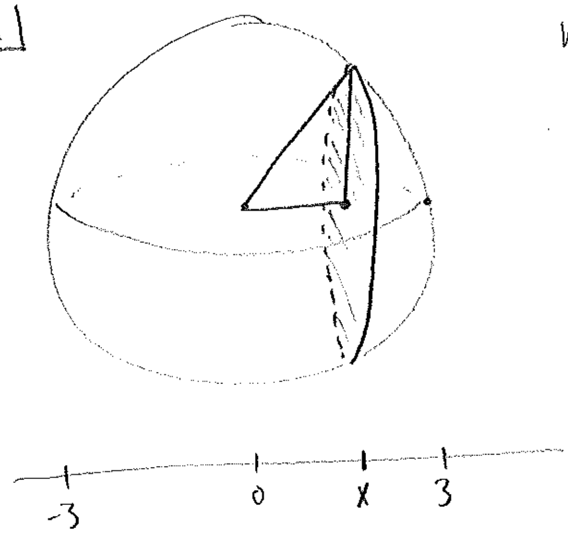
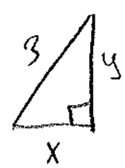
§6.2 Volume

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\, dy = \int_a^b A(x)\, dx$$

ex 1



need formula for $A(x)$.



$y = \sqrt{9 - x^2}$

$A(x) = \pi\left(\sqrt{9 - x^2}\right)^2 = \pi(9 - x^2)$

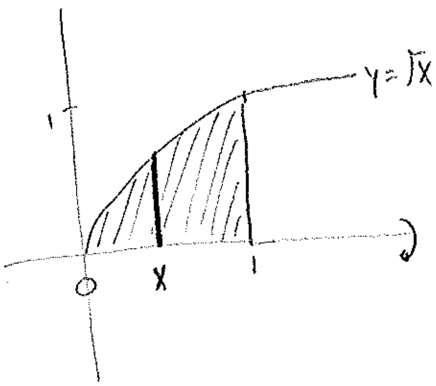
$\text{Vol} = \int_{-3}^{3} \pi(9 - x^2)\, dx$

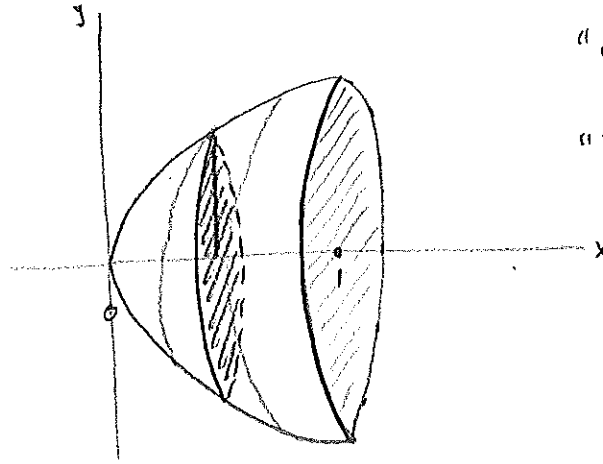
$$= \pi\left(9x - \frac{x^3}{3}\right)\Bigg|_{-3}^{3} = \pi\left(27 - \frac{27}{3}\right) - \pi\left(-27 + \frac{27}{3}\right)$$

$$= \pi\left(\frac{3\cdot 27}{3} - \frac{27}{3} + \frac{3\cdot 27}{3} - \frac{27}{3}\right) = \frac{4\cdot 27}{3}\pi$$

Solids of Revolution

ex) find volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



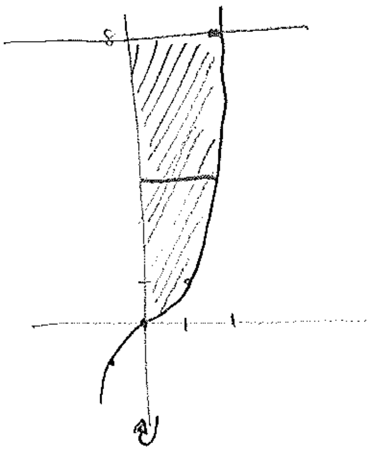


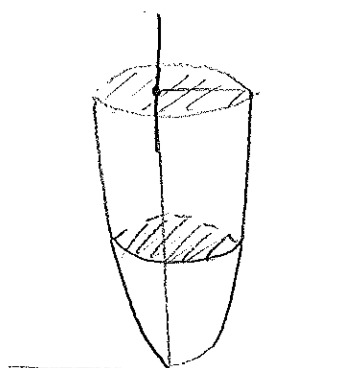
"acorn"

"x-mas ham"

$A(x) = \pi (\sqrt{x})^2 = \pi x$ $\qquad$ $\text{Vol} = \int_0^1 \pi x \, dx = \pi \frac{x^2}{2} \Big|_0^1 = \frac{\pi}{2}$

ex) find the volume of the solid obtained by rotating the region bounded by $y = x^3$ $\quad y = 8$ $\quad x = 0$ $\quad$ about the y-axis





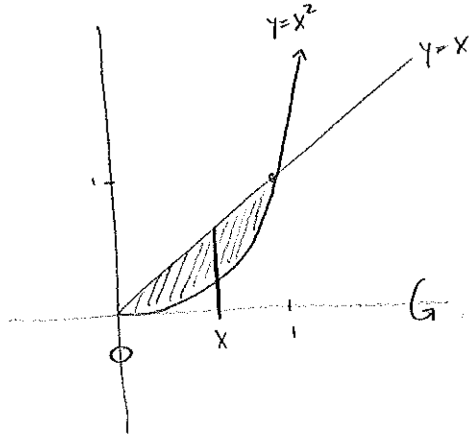
$A(y) = \pi \left(y^{1/3} \right)^2 = \pi y^{2/3}$

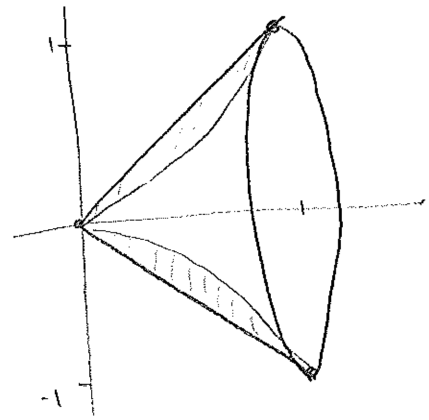
$\text{Vol} = \int_0^8 \pi y^{2/3} \, dy$

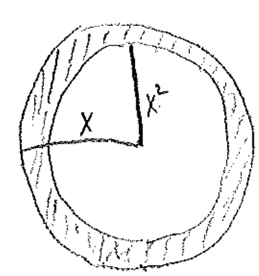
$= \pi \cdot \frac{3}{5} y^{5/3} \Big|_0^8 = \pi \cdot \frac{3}{5} \cdot 32$

Washer method

ex1 find the volume of the solid obtained by rotating the region enclosed by the curves $y=x$ and $y=x^2$ about the x-axis



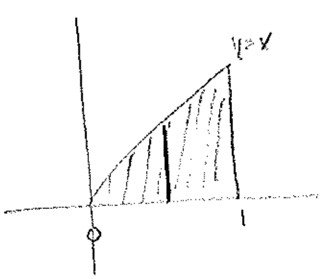


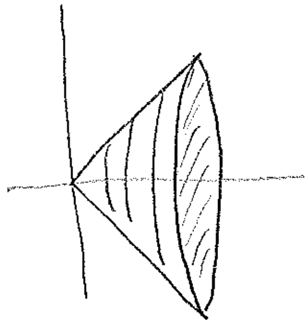


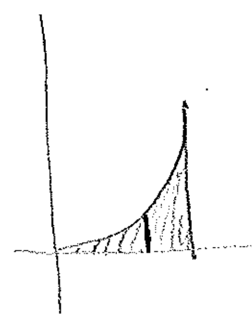
$$A(x) = \pi x^2 - \pi x^4$$

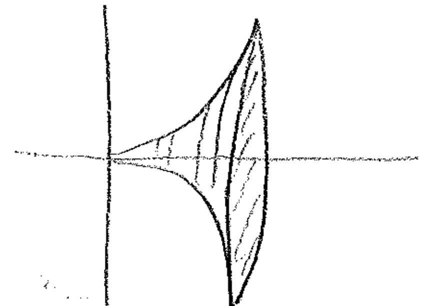
$$\text{Vol} = \int_0^1 \pi (x^2 - x^4)\, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

Another interpretation: We can find the volume by rotating $y=x$ and subtract the volume by rotating $y=x^2$





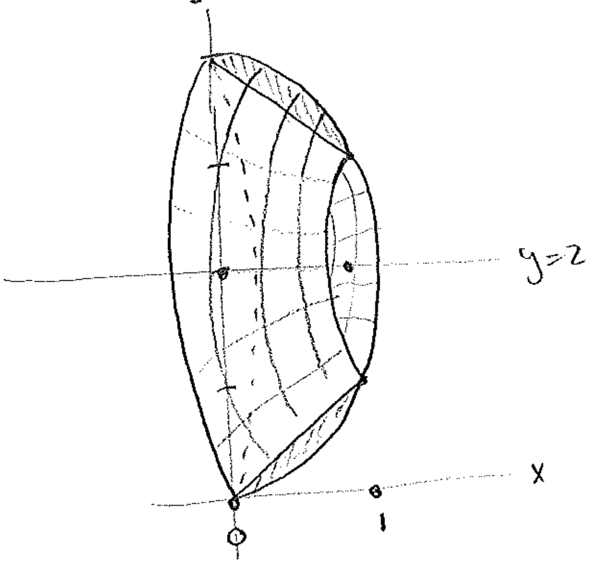


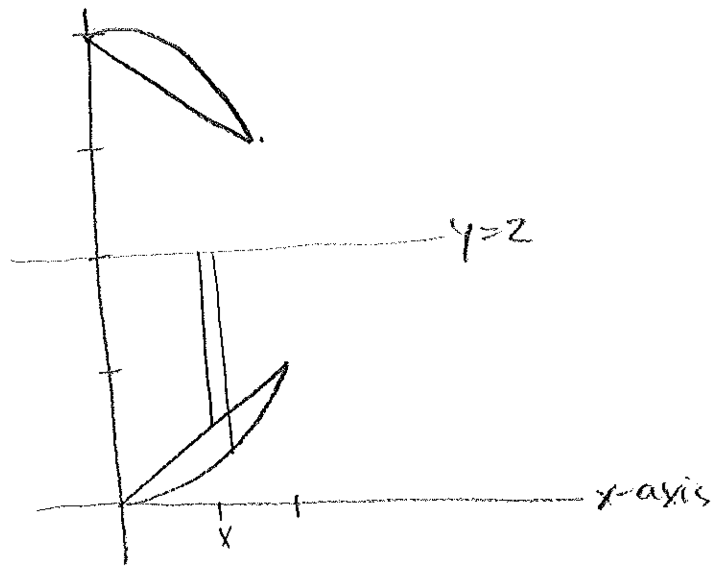


$$\text{Vol} = \int_0^1 \pi x^2 \, dx \qquad\qquad \text{Vol} = \int_0^1 \pi x^4 \, dx$$

hence difference $= \int_0^1 \pi x^2 dx - \int_0^1 \pi x^4 dx = \int_0^1 \pi (x^2 - x^4)\, dx$

ex] find volume obtained by rotating the same region (on p.3) about the line $y=2$.





inner radius : $2-x$ outer radius : $2-x^2$

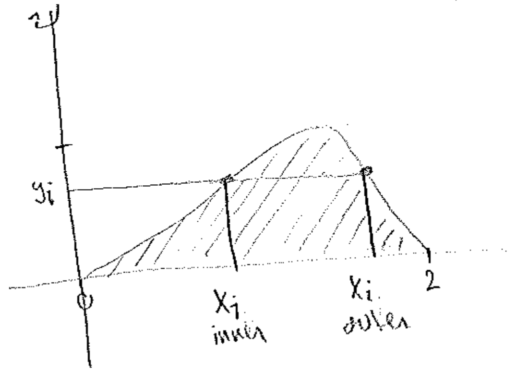
$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2 = \pi\left(4-4x^2+x^4-(4-4x+x^2)\right)$$

$$\text{Vol} = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^4-5x^2+4x)\,dx = \pi\left(\frac{x^5}{5} - \frac{5}{3}x^3 + 2x^2\right)\Bigg|_0^1$$

$$= \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \frac{8\pi}{15}$$

§6.3 Volume by Cylindrical Shells

ex)



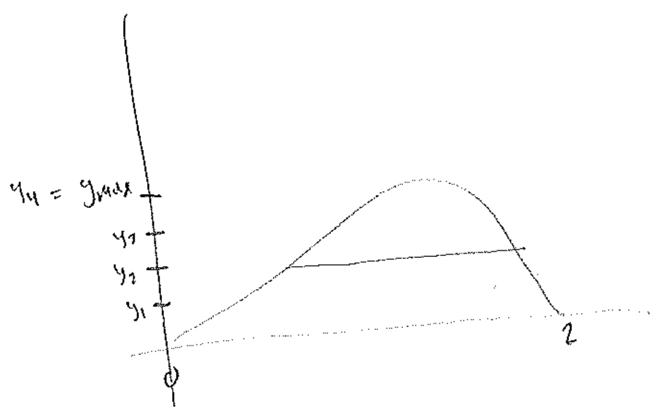
$y = 2x^2 - x^3$ and $y = 0$

rotate about the y-axis

Q: disk or washer? A: neither

Washer: $\int_0^{y_{max}} A(y)\,dy$ where $A(y_i) = \pi\left(x_{i_{outer}}^2 - x_{i_{inner}}^2\right)$

The washer method becomes very hard for this problem.

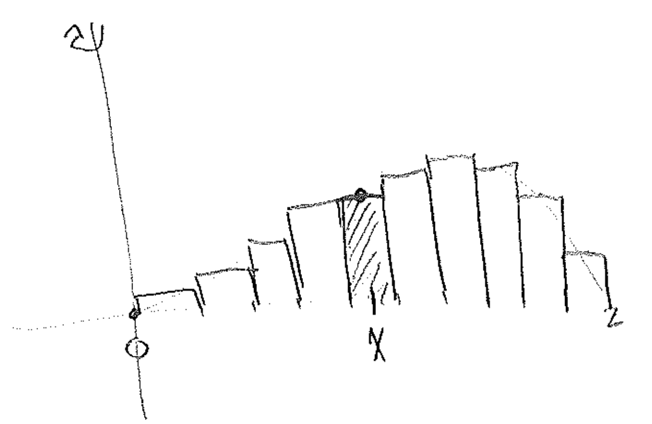


$$\Delta y = \frac{y_{max} - 0}{n}$$

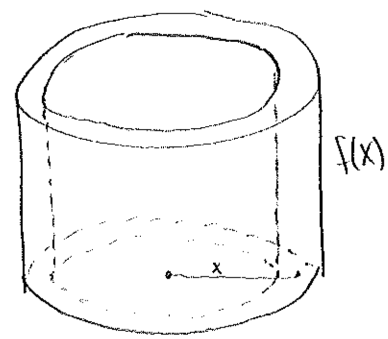
$$Vol = \lim_{n \to \infty} \sum_{i=1}^{n} A(y_i)\,\Delta y$$

area of i^{th} washer

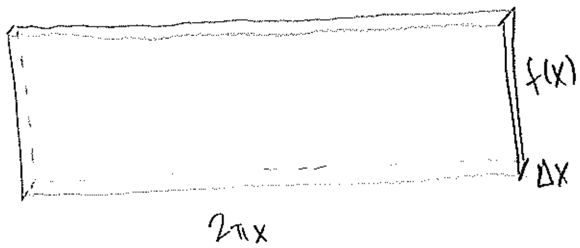
different method:



spin rectangle about y-axis yielding a cylindrical shell



now just lay the cylindrical shell flat

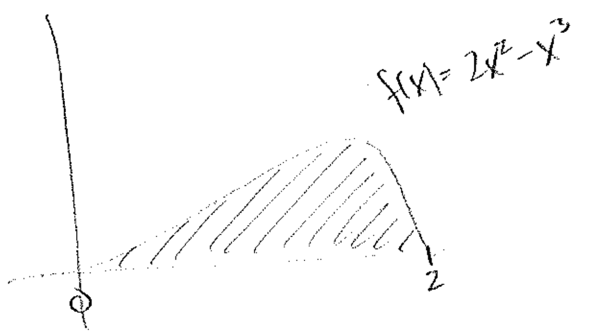


$\text{Vol} = 2\pi x \cdot f(x) \cdot \Delta x$

$$\boxed{\text{Vol} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x \cdot f(x) \cdot \Delta x = \int_a^b 2\pi x \cdot f(x)\, dx}$$

The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b.

ex



$$\text{Vol} = \int_0^2 2\pi x \cdot (2x^2 - x^3)\, dx = 2\pi \int_0^2 2x^3 - x^4\, dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5} \right) \Bigg|_0^2$$

$$= 2\pi \left(8 - \frac{32}{5} \right) = \frac{16}{5}\pi$$