Lecture Notes 1/22

Last time: Indefinite Integral vs. Definite Integral.

- Indefinite Integral is a function $\int x^2\,dx = \frac{x^3}{3} + C$

- Definite Integral is a number $\int_0^1 x^2\,dx = \frac{1}{3}$

Connection by FTC, 2 $\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b$

So to solve $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

Q: When do we need $+C$?

ex) $\int x(1-x)^2\,dx = \int x(1-2x+x^2)\,dx = \int x - 2x^2 + x^3\,dx$

$$= \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} + C$$

ex) $\int_1^2 x(1-x)^2\,dx = \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\Big|_1^2 = \frac{2^2}{2} - \frac{2}{3}2^3 + \frac{2^4}{4} - \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$

The distance / velocity / acceleration Problem

$v(t) = s'(t) \qquad a(t) = v'(t) = s''(t).$

Given initial velocity and acceleration for a particle moving on a straight line

$a(t) = t + 4 \quad , \quad v(0) = 5 \qquad 0 \leq t \leq 10$

a) find velocity at time t

b) find the displacement and total distance traveled

A: $a(t) = v'(t)$ means $v(t)$ is an antiderivative of $a(t)$

$v(t) = \int a(t)\,dt = \int t+4\,dt = \frac{1}{2}t^2 + 4t + C$

Initial condition $v(0) = 5$ thus $5 = v(0) = \frac{1}{2}0^2 + 4 \cdot 0 + C$

$5 = C$

so $v(t) = \frac{1}{2}t^2 + 4t + 5$

displacement $= s(10) - s(0) = \int_0^{10} v(t)\,dt = \int_0^{10} \frac{1}{2}t^2 + 4t + 5\,dt$

$= \frac{t^3}{6} + 2t^2 + 5t \Big|_0^{10} = \frac{1000}{6} + 2 \cdot 100 + 50 - 0 = 416\,2/3$

note: total distance $= \int_0^{10} |v(t)|\,dt = \int_0^{10} v(t)\,dt =$ displacement

in this example velocity is positive on interval $0 \leq t \leq 10$

so distance equals displacement.

§5.5 Substitution

Want to be able to differentiate more complicated functions.

$\int \sqrt{1+x^2} \cdot 2x \, dx$ $\qquad$ $u = 1+x^2$

$du = 2x \, dx$ $\qquad$ differential of u

then substitute u and its differential du into the integral

$$\int \sqrt{1+x^2} \cdot 2x \, dx = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

<u>Check</u>: $\frac{d}{dx}\left[\frac{2}{3}(1+x^2)^{3/2} + C\right] = \frac{2}{3} \cdot \frac{3}{2}(1+x^2)^{1/2}(2x) = \sqrt{1+x^2} \cdot 2x$

[Chain Rule]

note: u-substitution is chain rule in reverse

<u>ex</u> $\int x^3 \cdot \cos(x^4+2) \, dx$ $\qquad$ $u = x^4+2$ $\quad$ $du = 4x^3 \, dx$

$\frac{1}{4} du = x^3 \, dx$

$$= \int \cos u \cdot \frac{1}{4} du = \frac{1}{4}\int \cos u \, du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4+2) + C$$

<u>Check</u>: $\frac{d}{dx}\left[\frac{1}{4} \sin(x^4+2) + C\right] = \frac{1}{4} \cos(x^4+2) \cdot 4x^3 = x^3 \cdot \cos(x^4+2)$

Substitution Overview:

We change a complicated integral into a simpler integral by replacing x with a new variable u that is a function of x.

Main challenge is to think of an appropriate substitution.

- try to choose u to be some part of the integrand whose differential also occurs (up to a constant).

ex1 $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$ $\qquad u = 1-4x^2 \qquad du = -8x\,dx$

$\qquad -\frac{1}{8}\,du = x\,dx$

$$= \int -\frac{1}{8}\frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}\cdot 2u^{1/2} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

ex1 $\displaystyle\int \sec^2 x \cdot \tan x\,dx$ $\qquad u = \tan x \qquad du = \sec^2 x\,dx$

$$= \int u\,du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$$

ex1 $\displaystyle\int \sec^2 x \cdot \tan^3 x\,dx$ $\qquad$ same substitution

$$= \int u^3\,du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$$