Lecture Notes 1/14 : Fundamental Thm of Calculus (5.3)

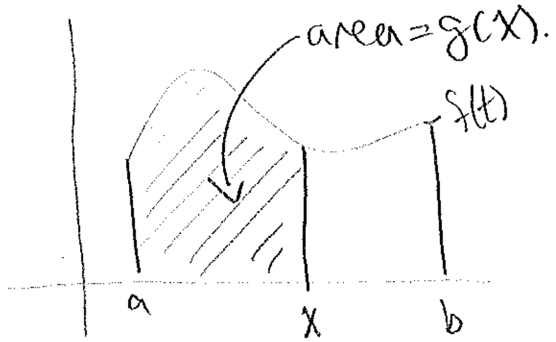
Brief History: tangent line problem → differential | Barrow, Newton | Trinity @ Cambridge

area problem → integral

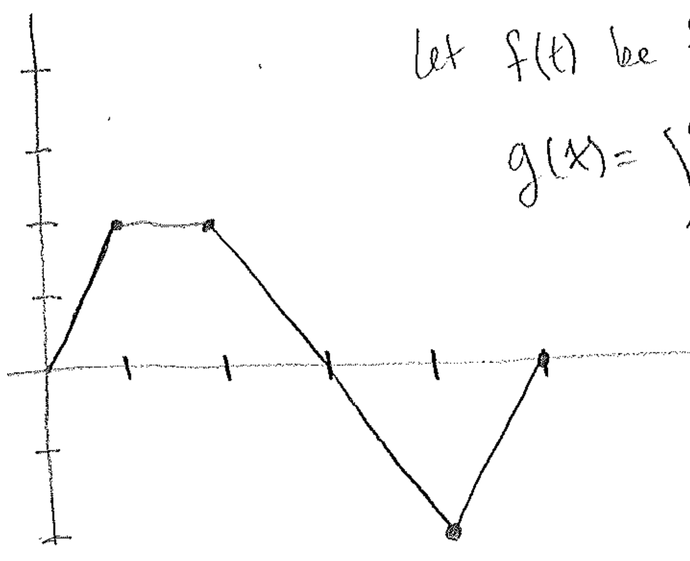
- differentiation and integration are inverse processes $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
- FTC gives this relationship. "if I have seen farther it is the live standing on the shoulders of giants"

"Area so far" function: $g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$

interpretation:



example:



Let $f(t)$ be the function shown and $g(x) = \int_0^x f(t)\,dt$. Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$

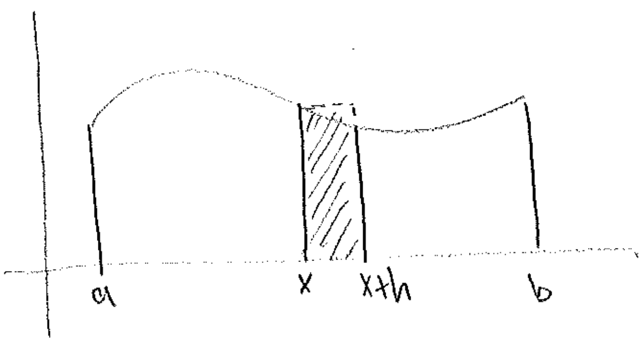
A: $g(0) = \int_0^0 f(t)\,dt = 0 \qquad g(1) = \int_0^1 f(t)\,dt = 1$

$g(2) = 3 \quad g(3) = 4 \quad g(4) = 3 \quad g(5) = 2$

Q: where does $g(x)$ achieve max? A: $x=3$

The main question: What is $g'(x)$?

know $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$



so for small h we have

$$g(x+h) - g(x) \approx h \cdot f(x)$$

Thus $g'(x) = \lim_{h\to 0} \frac{h \cdot f(x)}{h} = \lim_{h\to 0} f(x) = f(x)$

Thm (Fund. Thm of Calc): If f continuous on $[a,b]$ then the function $g(x) = \int_a^x f(t)\,dt$ $a \le x \le b$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$

Simplified: If $g(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$ then $g'(x) = f(x)$.

i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

note: this begins to illustrate the inverse processes of the derivative and the integral.

ex) If $g(x) = \int_0^x \sqrt{1+t^2}\, dt$ find $g'(x)$.

A: apply the thm and $g'(x) = \sqrt{1+x^2}$

ex) Find $\frac{d}{dx} \int_1^{x^4} \sec t\, dt$

A: Chain Rule and FTC will be used: let $u = x^4$

Then $\frac{d}{dx} \int_1^{x^4} \sec t\, dt = \frac{d}{dx} \int_1^{u} \sec t\, dt = \frac{d}{du}\left[\int_1^u \sec t\, dt\right] \cdot \frac{du}{dx}$

$= \sec u \cdot \frac{du}{dx} = \sec x^4 \cdot 4x^3$

notation: let $g(x) = \int_1^x \sec t\, dt$ and $u(x) = x^4$ so $g(u(x)) = \int_1^{x^4} \sec t\, dt$

then $\frac{d}{dx} \int_1^{x^4} \sec t\, dt = \frac{d}{dx} g(u(x)) = g'(u(x)) \cdot u'(x) = \sec(x^4) \cdot 4x^3$