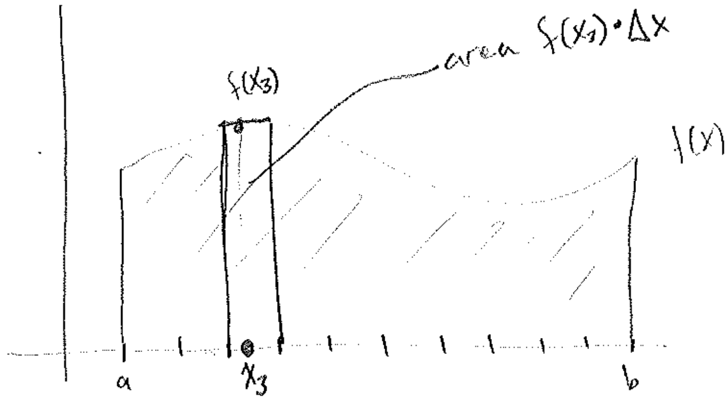
Last time:



Q: what is area below $f(x)$ between a and b ?

A: break interval $[a,b]$ into n subintervals of length $\frac{b-a}{n} = \Delta x$

In each subinterval choose a sample point x_i so that

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{and} \quad \text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

we call this the RIEMANN SUM for $f(x)$ between a and b

Recall: we can use left / right / mid / random points ... does not matter what our sample point is b/c in the limit the sum is equivalent.

Def for f a continuous function, we define the definite integral of f from a to b by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

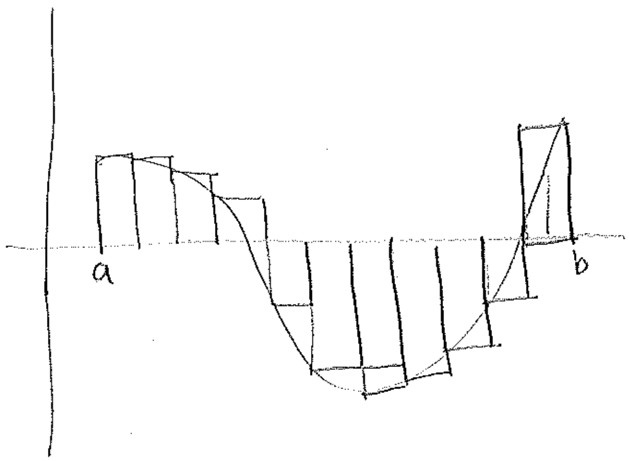
note: the definite integral is a number

Interpretation of the Definite Integral

For $f(x) \geq 0$ (positive) we know that $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ is the area under the curve from a to b.

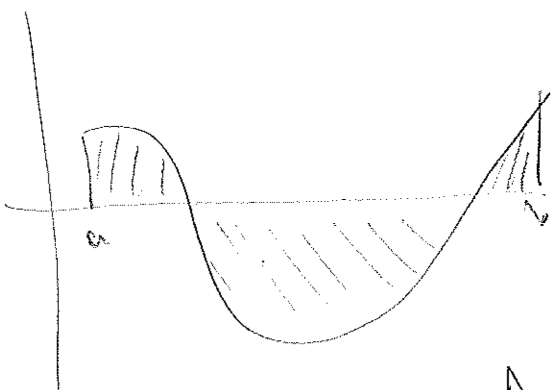
Thus for $f(x) \geq 0$ we have $\int_a^b f(x)\,dx$ = area under curve $f(x)$ from a to b.

What if $f(x)$ takes both positive and negative values?



Then the sum $\sum_{i=1}^{n} f(x_i)\Delta x$ will be the sum of the areas of rectangles above the x-axis, with the negatives of the area of the rectangles below the x-axis.

In the limit we will have the image below. Hence the definite integral can be interpreted as a net area, i.e. the difference of two areas



$$\int_a^b f(x)dx = A_1 - A_2$$ where

A_1 is the area of the region above the x-axis and below the graph and A_2 is the Area of the region below the x-axis and above the graph.

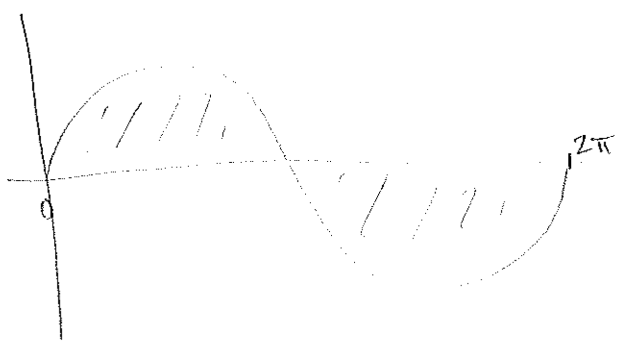
<u>examples</u>: 1) express $\lim_{n\to\infty} \sum_{i=1}^{n} [x_i^3 + x_i \sin(x_i)] \Delta x$ as definite integral on the interval $[0, \pi]$

A: let $f(x) = x^3 + x \sin x$. then the definite integral is

$$\int_0^{\pi} (x^3 + x \sin x)\, dx$$

2) Evaluate $\int_0^{2\pi} \sin x \, dx$

A:

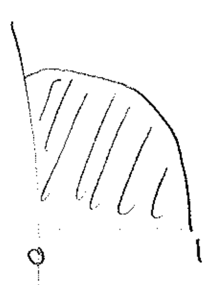


$\rightsquigarrow 0.$

3) Evaluate $\int_0^1 \sqrt{1-x^2}\, dx$

well $x^2+y^2=1$ is unit circle so graph is

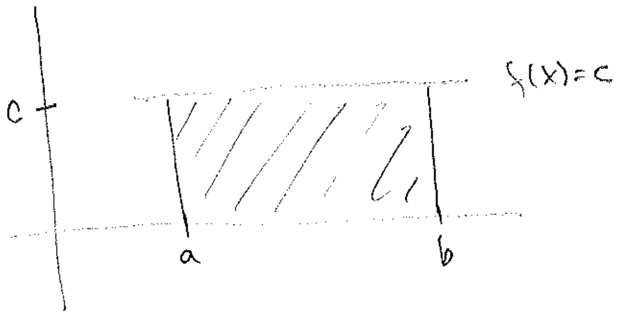
with $\sqrt{1-x^2} \geq 0$ hence



$$\int_0^1 \sqrt{1-x^2}\, dx = \text{Area} = \frac{\pi}{4}$$

Properties of the Definite Integral (pg 379)

1. $\int_a^b c\,dx = c(b-a)$ where c is any constant.



Area $= c\cdot(b-a)$

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Similar to $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

3. $\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$ where c is any constant

Intuitively, multiplying the function by c will multiply each approximating rectangle by c, hence the entire area/integral.

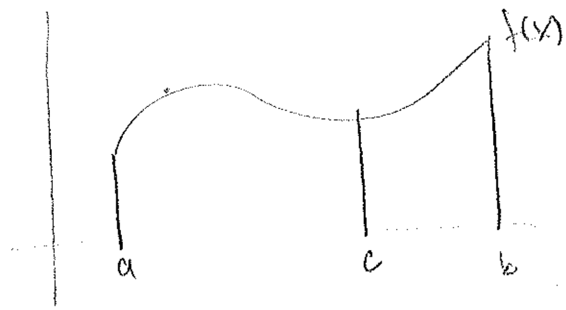
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

follows from properties 2 and 3 with $c=-1$.

note: these properties hold for any $f(x)$, not just $f(x)\geq 0$. However our geometric interpretation makes the most sense for positive functions.

5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Geometrically



ex] Given $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$. Find $\int_8^{10} f(x)\,dx$.

By (5) know $\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$

hence $\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$

6. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

Interpretation: we know areas are positive.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Bigger functions have bigger integrals.

Follows from 6 using the fact $f(x) - g(x) \geq 0$

Note: we will omit property (8).

0. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

ex) Recall that $\int_0^1 x^2\,dx = 1/3$

use this fact to evaluate $\int_0^1 (5-6x^2)\,dx$

A: 3

ex) Write $\int_{-2}^{2} f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$

as a single integral in the form $\int_a^b f(x)\,dx$

A: $\int_{-1}^{5} f(y)\,dy$