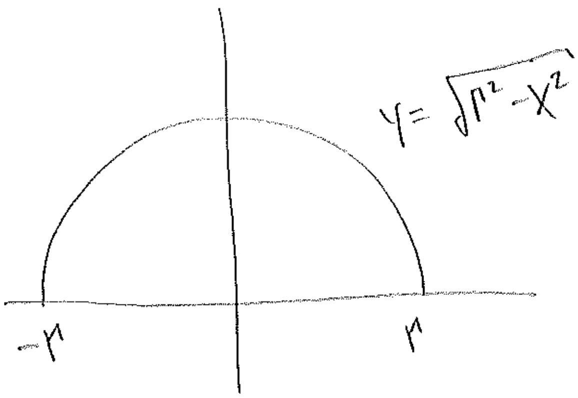
HW 8 Solutions

1. Arc length of circle with equation $x^2+y^2=r^2$

note: we will calculate length of top half of circle and multiply by 2.



$$y' = \frac{-2x}{2\sqrt{r^2-x^2}} \quad \text{via chain rule}$$

$$(y')^2 = \frac{x^2}{r^2-x^2}$$

$$\text{length} = 2\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\int_{-r}^{r}\sqrt{\frac{r^2-x^2}{r^2-x^2}+\frac{x^2}{r^2-x^2}}\,dx$$

$$=2\int_{-r}^{r}\sqrt{\frac{r^2}{r^2-x^2}}\,dx = 2\int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}\,dx \qquad \begin{array}{l} x=r\sin\theta \\ dx=r\cos\theta\,d\theta \end{array}$$

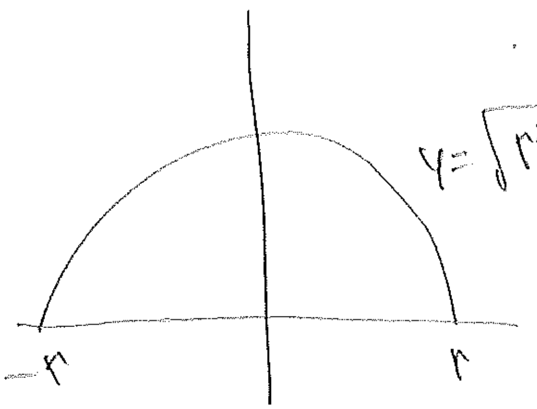
$$=2\int\frac{r}{r\cos\theta}\cdot r\cos\theta\,d\theta = 2\int r\,d\theta = 2r\theta\Big|_{x=-r}^{x=r}$$

$$=2r\cdot\arcsin\left(\frac{x}{r}\right)\Big|_{-r}^{r} = 2r\cdot\arcsin(1)-2r\cdot\arcsin(-1)$$

$$=2r\left(\frac{\pi}{2}\right)-2r\left(-\frac{\pi}{2}\right) = 2r\cdot\pi$$

2. Surface area of sphere with radius r.

We rotate the top half of a circle about the x-axis



$$y' = \frac{-2x}{2\sqrt{r^2 - x^2}}$$

$$(y')^2 = \frac{x^2}{r^2 - x^2}$$

$$SA = \int_{-r}^{r} 2\pi \sqrt{r^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= \int_{-r}^{r} 2\pi \sqrt{r^2 - x^2 + x^2}\, dx = \int_{-r}^{r} 2\pi \sqrt{r^2}\, dx$$

$$= \int_{-r}^{r} 2\pi r\, dx = 2\pi r x \Big|_{-r}^{r} = 2\pi r^2 + 2\pi r^2 = 4\pi r^2$$

8.1.1 $y = 2x - 5 \quad y' = 2 \quad (y')^2 = 4$

$$\text{length} = \int_{-1}^{3} \sqrt{1+4}\,dx = x \cdot \sqrt{5}\Big|_{-1}^{3} = 4\sqrt{5}$$

Check: $f(-1) = -7 \quad f(3) = 1$

thus the two points are $(-1,-7)$ and $(3,1)$

$$\text{distance} = \sqrt{(3+1)^2 + (1+7)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$$

8.1.8 $y = 2(x+4)^{3/2} \quad y' = 3(x+4)^{1/2} \quad (y')^2 = 9(x+4)$

$$\text{length} = \int_0^2 \sqrt{1+9x+36}\,dx = \frac{2}{3}(9x+37)^{3/2} \cdot \frac{1}{9}\Big|_0^2$$

$$= \frac{2}{3}(18+37)^{3/2} \cdot \frac{1}{9} - \frac{2}{3}(37)^{3/2} \cdot \frac{1}{9}$$

8.1.12 $y = \ln(\cos x)$ $\quad y' = \dfrac{-\sin x}{\cos x}$ via chain rule

$(y')^2 = \tan^2 x$

$$\text{length} = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/3} \sec x\, dx = \ln(\sec x + \tan x)\Big|_0^{\pi/3}$$

$$= \ln\left(\sec(\pi/3) + \tan(\pi/3)\right) - \ln(\sec(0) + \tan(0))$$

8.1.35 $y = \arcsin x + \sqrt{1-x^2}$

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{-2x}{2\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \qquad (y')^2 = \frac{(1-x)^2}{1-x^2}$$

$$s(x) = \int_0^x \sqrt{1+\frac{(1-t)^2}{1-t^2}}\, dt = \int_0^x \sqrt{\frac{1-t^2}{1-t^2} + \frac{1-2t+t^2}{1-t^2}}\, dt$$

$$= \int_0^x \sqrt{\frac{2-2t}{1-t^2}}\, dt = \int_0^x \sqrt{\frac{2\cdot(1-t)}{(1-t)(1+t)}}\, dt = \int_0^x \frac{\sqrt{2}}{\sqrt{1+t}}\, dt$$

$$= \sqrt{2}\cdot\sqrt{1+t}\cdot 2\Big|_0^x = 2\sqrt{2}\cdot\sqrt{1+x} - 2\sqrt{2}$$

8.1.41 $y = \int_1^x \sqrt{t^3-1}\,dt$

$y' = \sqrt{x^3-1}$ via FTC, Part 1.

$(y')^2 = x^3-1$

$\text{length} = \int_1^4 \sqrt{1+x^3-1}\,dx = \int_1^4 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big|_1^4$

$= \frac{2}{5}4^{5/2} - \frac{2}{5}$

8.2.2a $y = x^{-2}$ $1 \le x \le 2$ $y' = -2x^{-3}$ $(y')^2 = 4x^{-6}$

i) $SA = \int_1^2 2\pi \cdot x^{-2} \cdot \sqrt{1+4x^{-6}}\,dx$

ii) $SA = \int_1^2 2\pi \cdot x\sqrt{1+4x^{-6}}\,dx$

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8.2.10 $y = \frac{x^3}{6} + \frac{1}{2x}$ $y' = \frac{x^2}{2} - \frac{1}{2x^2}$

$(y')^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^2}$

$$\text{length} = \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$$

$$= \int_{1/2}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$= 2\pi \int_{1/2}^{1} \frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\, dx$$

$$= 2\pi\, \frac{x^6}{72} + \frac{x^2}{24} + \frac{x^2}{8} - \frac{1}{8x^2} \Big|_{1/2}^{1}$$

$$= 2\pi \left[\frac{1}{72} + \frac{1}{24} + \frac{1}{8} - \frac{1}{8}\right] - 2\pi \left[\frac{1}{72 \cdot 2^6} + \frac{1}{24 \cdot 4} + \frac{1}{8 \cdot 4} - \frac{4}{8}\right]$$

<u>8.2.14</u> $y = 1 - x^2$ $y' = -2x$ $(y')^2 = 4x^2$

$$SA = \int_0^1 2\pi x \sqrt{1+4x^2}\, dx \qquad \begin{array}{l} u = 1+4x^2 \\ du = 8x\,dx \end{array}$$

$$= \frac{1}{8}\int_0^1 2\pi \sqrt{u}\, du = \frac{1}{8}\cdot\frac{2}{3}\cdot 2\pi \cdot u^{3/2} \Big|_{x=0}^{x=1}$$

$$= \frac{\pi}{6}(1+4x^2)^{3/2}\Big|_0^1 = \frac{\pi}{6}(5)^{3/2} - \frac{\pi}{6}$$