Math 2 Winter 2013
Optional Extra Credit Project

Successful completion of this project will allow you to drop your lowest quiz grade of the term.

THE PROJECT:

Find a real-world application of calculus. This "real-world application" can be from another class, a textbook from another subject, a news article, or something else entirely. It should not come directly from our textbook.

You have two options for the format of the assignment. It can be either a 1.5 to 2 page paper or a 5 to 6 minute informal presentation in your professor's office. Your paper or presentation should include the following:

An explanation of the real world setting.
An explanation of what aspects of calculus are involved and where they are used.
A solution of the real world problem using calculus.

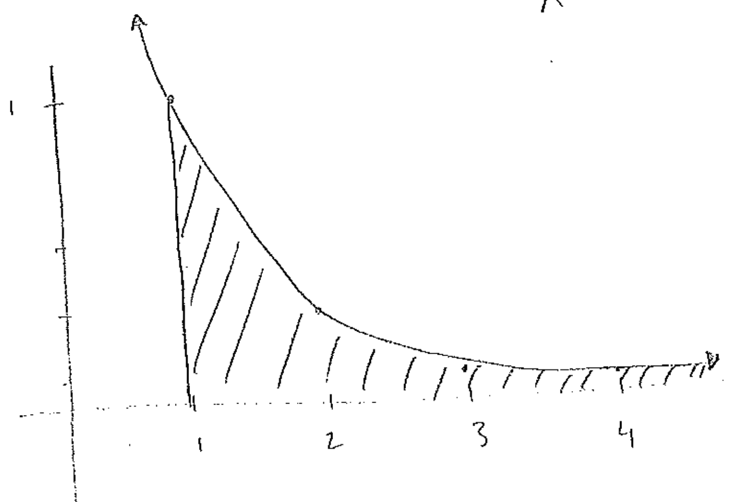
Please email your professor with your choice of topic and whether you will be doing a paper or a presentation by **Friday March 1.**

If you are writing a paper it will be due **Friday March 8**, the last day of class. Presentations will be scheduled during the last week of classes.

§7.8 Improper Integrals

$$\int_a^\infty f(x)\,dx \qquad \int_{-\infty}^a f(x)\,dx \qquad \int_{-\infty}^\infty f(x)\,dx$$

Motivating Example $\quad f(x) = \dfrac{1}{x^2}$



Q: what is the area below the curve, above the x-axis and to the right of the line $x=1$, i.e. $\int_1^\infty \frac{1}{x^2}\,dx$

Guess? ⟶ Infinite?

$$\int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^2 = \frac{-1}{2} - (-1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\int_1^3 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^3 = \frac{-1}{3} - (-1) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_1^{10} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{10} = \frac{-1}{10} + 1 = \frac{9}{10}$$

Q: What is the area between $x=1$ and $x=t$?

$$\int_1^{t} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{t} = \frac{-1}{t} + 1 = 1 - \frac{1}{t}$$

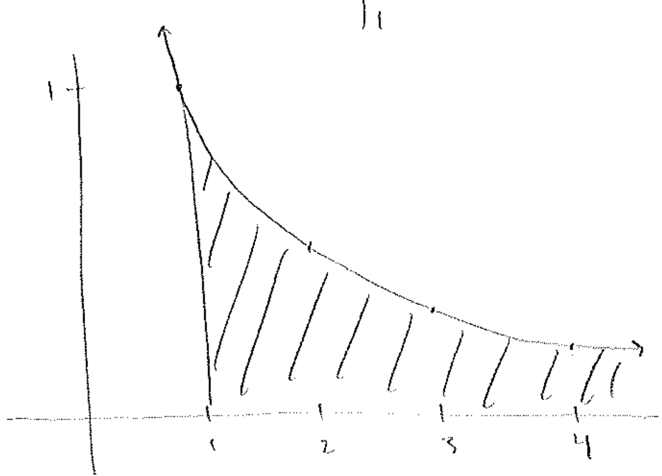
What happens when t gets very large? 17 billion?

Guess again for $\int_1^{\infty} \frac{1}{x^2}\,dx$?

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 - \frac{1}{\infty} = 1 - 0 = 1$$

Whoa! this seemingly infinite area is actually equal to 1.

Next example: $\int_1^{\infty} \frac{1}{x}\,dx$



Q: what is the area below the curve, above the x-axis and to the right of the line $x=1$

Q: is the area greater OR less than 1 ?

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\Big|_1^t = \lim_{t\to\infty} \ln(t) = \infty$$

i.e. the limit does NOT exist as a finite number.

We say the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges

def: The improper integral $\int_a^\infty f(x)\,dx$ converges (OR is convergent)

if the limit $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists and diverges (OR is divergent) if the

limit does NOT exist as a finite number.

Thus $\int_1^\infty \frac{1}{x^2}\,dx$ converges and $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Let's compare $\frac{1}{x^2}$ vs. $\frac{1}{x}$. They have similar graphs, but $\frac{1}{x^2}$

"dies," OR "goes to zero" faster than $\frac{1}{x}$.

Q: for what values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

$p>2$? $p<1$?

A: FACT: $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$, diverges for $p\leq 1$

In general, to evaluate improper integrals

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \qquad \int_{-\infty}^a f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx$$

and if both the above integrals converge we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

ex) $$\int_3^\infty \frac{1}{(x-2)^3}dx = \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^3}dx = \lim_{t\to\infty} \frac{-1}{2}\cdot\frac{1}{(x-2)^2}\Big|_3^t$$

$$= \lim_{t\to\infty} \frac{-1}{2(t-2)^2} + \frac{1}{2(3-2)^2} = \frac{1}{2}$$

this example 2/27

ex) $$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx \qquad \begin{array}{ll} u=x & dv=e^x dx \\ du=dx & v=e^x \end{array}$$

$$\int_t^0 xe^x\,dx = xe^x - \int e^x dx = xe^x - e^x\Big|_t^0 = -1-(te^t-e^t)$$

but $\lim_{t\to-\infty} -1-te^t+e^t = -1 - \frac{-\infty}{\infty} + \frac{1}{\infty}$ indeterminant form

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} \overset{\text{[L'Hopital's Rule]}}{=} \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \frac{1}{-\infty} = 0$$

Thus $$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} -1-te^t+e^t = -1+0+0 = -1$$

ex1 $\displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to-\infty} \int_{t}^{0} (1-x)^{-1/2}\,dx = \lim_{t\to-\infty} -2(1-x)^{1/2}\Big|_{t}^{0}$

$\displaystyle= \lim_{t\to-\infty} -2 - \left(-2\sqrt{1-t}\right) = \lim_{t\to-\infty} -2+2\sqrt{1-t} = \infty$

divergent.