Last time: Partial Fractions

Case 1: $Q(x)$ is a product of non-repeating linear factors

ex) $$\int \frac{x-4}{x^2+3x+2}\,dx$$

Step 1: factor $Q(x) = x^2+3x+2 = (x+2)(x+1)$

Step 2: determine the constants A and B such that

$$\frac{x-4}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$x-4 = A(x+1) + B(x+2)$$

$$x-4 = (A+B)x + (A+2B)\cdot 1$$

$1 = A+B$ $\qquad -4 = A+2B$

$1-B = A$ $\qquad -4 = 1-B+2B$

$6 = A$ $\qquad -5 = B$

$$\int \frac{x-4}{x^2+3x+2}\,dx = \int \frac{6}{x+2} - \frac{5}{x+1}\,dx = 6\ln(x+2) - 5\ln(x+1) + C$$

Case 2: $Q(x)$ is product of linear factors, some of which are repeated.

ex] $\displaystyle\int \frac{4x}{x^3-x^2-x+1}$ $\qquad Q(x) = (x-1)(x^2-1) = (x-1)(x-1)(x+1)$

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2.$$

$$4x = A(x^2-1) + B(x+1) + C(x^2-2x+1)$$

$$4x = (A+C)x^2 + (B-2C)x + (-A+B+C)\cdot 1$$

$A+C=0$ $\qquad B-2C=4$ $\qquad -A+B+C=0$

$A=-C$ $\qquad B+B=4$ $\qquad 2C+B=0$

$A=1$ $\qquad B=2$ $\qquad B=-2C$

$C=-1$

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,dx$$

$$= \ln(x-1) - \frac{2}{x-1} - \ln(x+1) + C$$

note: can use $\begin{array}{l} u = x-1 \\ du = dx \end{array}$ $\quad \displaystyle\int \frac{2}{u^2}\,du = -2u^{-1} = \frac{-2}{x-1}$

ex1 $\displaystyle\int \frac{x^2-2}{x^3+2x^2+x}\,dx$ $\qquad Q(x) = x(x^2+2x+1) = x(x+1)^2$

$$\frac{x^2-2}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2-2 = A(x^2+2x+1) + Bx(x+1) + Cx$$

$$x^2-2 = A(x^2+2x+1) + B(x^2+x) + C(x)$$

$$x^2-2 = (A+B)x^2 + (2A+B+C)x + A\cdot 1$$

$A+B=1 \qquad 2A+B+C=0 \qquad A=-2$

$B=3 \qquad -4+3+C=0$

$C=1$

$$\int \frac{x^2-2}{x^3+2x^2+x}\,dx = \int \frac{-2}{x} + \frac{3}{x+1} + \frac{1}{(x+1)^2}\,dx$$

$$= -2\ln(x) + 3\ln(x+1) - \frac{1}{x+1} + C$$

Case 3: $Q(x)$ contains non-repeated irreducible quadratic factors,

Case 4: $Q(x)$ contains a repeated irreducible quadratic factor

note: an irreducible quadratic is x^2+1, for example