Integration of Rational Functions by Partial Fractions

Ex 1 $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx$

idea: rewrite the integrand

$$\frac{x+5}{x^2+x-2} = \frac{2x+4-(x-1)}{(x+2)(x-1)} = \frac{2x+4}{(x+2)(x-1)} - \frac{(x-1)}{(x+2)(x-1)}$$

$$= \frac{2(x+2)}{(x+2)(x-1)} - \frac{(x-1)}{(x+2)(x-1)} = \frac{2}{x-1} - \frac{1}{x+2}$$

so that $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{2}{x-1} - \frac{1}{x+2}\,dx = 2\ln(x-1) - \ln(x+2) + C$

Rational Functions: $f(x) = \dfrac{P(x)}{Q(x)}$

we will always have $\deg P(x) < \deg Q(x)$

we omit the method when $\deg P(x) \geq \deg Q(x)$

- Step 1: factor denominator $Q(x)$.

note: our denominators will always factor into a product of linear factors $(ax+b)$ (ie. NOT ax^2+bx+c)

we omit the method when this does not happen (Case 3 + 4)

- Step 2: rewrite the rational function $\frac{P(x)}{Q(x)}$ as a sum of partial fractions of the form $\frac{A}{(ax+b)^n}$

Case 1: $Q(x)$ is a product of non-repeating linear factors.

FACT: If $Q(x)$ has (3) linear factors then there exist (3) constants A, B, C such that

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3}$$

ex1 $\displaystyle\int \frac{x}{x^2+x-2}\,dx$ $\qquad Q(x) = (x+2)(x-1)$

then there exist constants A and B such that

$$\frac{x}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$x = A(x-1) + B(x+2)$$

$$x + 0 = (A+B)x + (-A+2B)\cdot 1$$

$1 = A+B$ $\qquad 0 = -A+2B$

$1 = 3B$ $\qquad A = 2B$

$B = 1/3$ $\qquad A = 2/3$

$$\int \frac{x}{x^2+x-2}\,dx = \int \frac{2/3}{x+2} + \frac{1/3}{x-1}\,dx$$

$$= \tfrac{2}{3}\ln(x+2) + \tfrac{1}{3}\ln(x-1) + C$$

ex1 $\displaystyle\int \frac{x+3}{x^2+6x+8}\,dx$

$x^2+6x+8 = (x+2)(x+4)$

$$\frac{x+3}{x^2+6x+8} = \frac{A}{x+2} + \frac{B}{x+4}$$

$x+3 = A(x+4) + B(x+2)$

$x+3 = (A+B)x + (4A+2B)\cdot 1$

$A+B=1 \qquad 4A+2B=3$

$A=1-B \qquad 4(1-B)+2B=3$

$4-4B+2B=3$

$A=1/2 \qquad -2B=-1$

$B=1/2$

$$\int \frac{x+3}{x^2+6x+8}\,dx = \int \frac{1/2}{x+2} + \frac{1/2}{x+4}\,dx = \frac{1}{2}\ln(x+2) + \frac{1}{2}\ln(x+4) + C$$

ex 1 $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$Q(x) = x(2x^2+3x-2) = x(2x-1)(x+2)$

thus $\displaystyle\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$

$x^2+2x-1 = A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$

$x^2+2x-1 = (2A+B+2C)x^2 + (3A+2B-C)x + (-2A)\cdot 1$

$2A+B+2C=1$ $\quad$ $3A+2B-C=2$ $\quad$ $-2A=-1$

$A = 1/2$

$B+2C=0$ $\quad$ $\frac{3}{2}-4C-C=2$

$B=-2C$ $\quad$ $-5C=\frac{1}{2}$

$B=1/5$ $\quad$ $C=-1/10$

$\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{1/2}{x} + \frac{1/5}{2x-1} - \frac{1/10}{x+2}\,dx$

$= \frac{1}{2}\ln(x) + \frac{1}{5}\cdot\frac{1}{2}\ln(2x-1) - \frac{1}{10}\ln(x+2) + C$