ntegration by Parts

$$\int u\,dv = uv - \int v\,du$$

Ex) $\displaystyle\int_1^3 x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{x^4}{4}$

$$= \frac{x^4}{4}\cdot \ln x - \int_1^3 \frac{x^3}{4}\,dx = \frac{x^4}{4}\ln x - \frac{1}{4}\,\frac{x^4}{4}\Bigg|_1^3 = \frac{3^4}{4}\ln 3 - \frac{1}{4}\cdot\frac{3^4}{4} + \frac{1}{16}$$

Ex) $\displaystyle\int e^x \sin x \, dx$

note: neither becomes simpler

$u = e^x \qquad dv = \sin x\,dx$

$du = e^x\,dx \qquad v = -\cos x$

$$\int e^x \sin x\,dx = -e^x\cos x + \int \cos x\, e^x\,dx$$

$u = e^x \qquad dv = \cos x\,dx$

$du = e^x\,dx \qquad v = \sin x$

$$= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$$

so we have $\displaystyle\int e^x \sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x \sin x\,dx$

$$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x$$

$$\int e^x \sin x\,dx = \frac{1}{2}\left(-e^x\cos x + e^x \sin x\right) + C$$

§7.2 Trig Integrals

In this section we learn how to integrate:

$$\int \sin^m x \cdot \cos^n x \, dx \quad \text{and} \quad \int \tan^m x \cdot \sec^n x \, dx$$

What does $\cos^n x$ mean?

Let $n=3$.

$$\cos^3 x = \cos^3(x) = (\cos x)^3 = \cos x \cdot \cos x \cdot \cos x$$

$$\neq \cos x^3 = \cos(x^3) = \cos(x \cdot x \cdot x)$$

method: eventually use u-sub, but first need to rewrite the integrand using the equalities $\cos^2 x + \sin^2 x = 1$ and $\sec^2 x = 1 + \tan^2 x$

ex1 $\int \cos^3 x \, dx$

try u-sub: $u = \cos x$ $du = -\sin x \, dx$ --- does NOT work.

idea: save one factor of $\cos x$ and rewrite the rest using

$$\cos^2 x + \sin^2 x = 1$$

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$ $du = \cos x \, dx$

$$= \int 1 - u^2 \, du = u - \frac{u^3}{3}$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

ex)

$$\int \sin^5 x \cdot \cos^2 x \, dx = \int \sin x \left(1-\cos^2 x\right)^2 \cdot \cos^2 x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2)^2 u^2 \, du$$

$$= -\int (1 - 2u^2 + u^4) u^2 \, du = -\int u^2 - 2u^4 + u^6 \, du$$

$$= -\left(\frac{u^3}{3} - \frac{2}{5} u^5 + \frac{1}{7} u^7 \right) + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

ex) $\int \tan^6 x \cdot \sec^4 x \, dx = \int \tan^6 x \cdot \sec^2 x \cdot \sec^2 x \, dx$

$$= \int \tan^6 x \, (1 + \tan^2 x) \sec^2 x \, dx \qquad u = \tan x \quad du = \sec^2 x \, dx$$

$$= \int u^6 (1 + u^2) \, du = \frac{1}{7} u^7 + \frac{1}{9} u^9 + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

ex) $\int \tan^5 x \sec^7 x \, dx = \int \tan^4 x \cdot \sec^6 x \cdot \sec x \tan x \, dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

$$= \int (\sec^2 x - 1)^2 \cdot \sec^6 x \cdot \sec x \tan x \, dx = \int (u^2-1)^2 u^6 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du = \int u^{10} - 2u^8 + u^6 \, du = \frac{u^{11}}{11} - \frac{2}{9} u^9 + \frac{1}{7} u^7 + C = \frac{\sec^{11} x}{11} - \frac{2}{9}\sec^9 x + \frac{\sec^7 x}{7} + C$$

<u>In general</u>:

$$\int \sin^m x \cos^n x \, dx$$

If either m or n is odd then save an factor of $\sin x$ or $\cos x$, respectively, and rewrite using $\cos^2 x + \sin^2 x = 1$
then substitute $u = \cos x$ or $u = \sin x$ respectively

note: if both even, then use $\frac{1}{2}$ angle formulas

$$\int \tan^m x \sec^n x \, dx$$

If n even, save $\sec^2 x$ and set $u = \tan x$ $du = \sec^2 x$

If m odd, save $\tan x \sec x$ and set $u = \sec x$
$du = \tan x \sec x$

rewrite using $\sec^2 x = 1 + \tan^2 x$