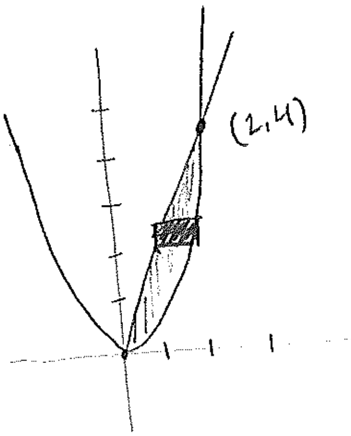
Volumes:

ex) find volume of the solid obtained by rotating about the x-axis the region bounded by $y=2x$ and $y=x^2$.



Washer: inner : x^2

outer : $2x$

$$A(x) = \pi(2x)^2 - \pi(x^2)^2 = \pi(4x^2 - x^4)$$

$$Vol = \int_0^2 \pi(4x^2 - x^4)\,dx = \pi\left(\tfrac{4}{3}x^3 - \tfrac{x^5}{5}\right)\Big|_0^2 = \frac{64}{15}\pi$$

Cylindrical Shells: $Vol = \int_0^4 2\pi y \cdot \left(\sqrt{y} - \tfrac{1}{2}y\right) dy = 2\pi \int_0^4 y^{3/2} - \tfrac{1}{2}y^2\,dy$

$$= 2\pi\left(\tfrac{2}{5}y^{5/2} - \tfrac{1}{6}y^3\right)\Big|_0^4 = 2\pi\left(\tfrac{2}{5}\cdot 32 - \tfrac{64}{6}\right) = \frac{64}{15}\pi$$

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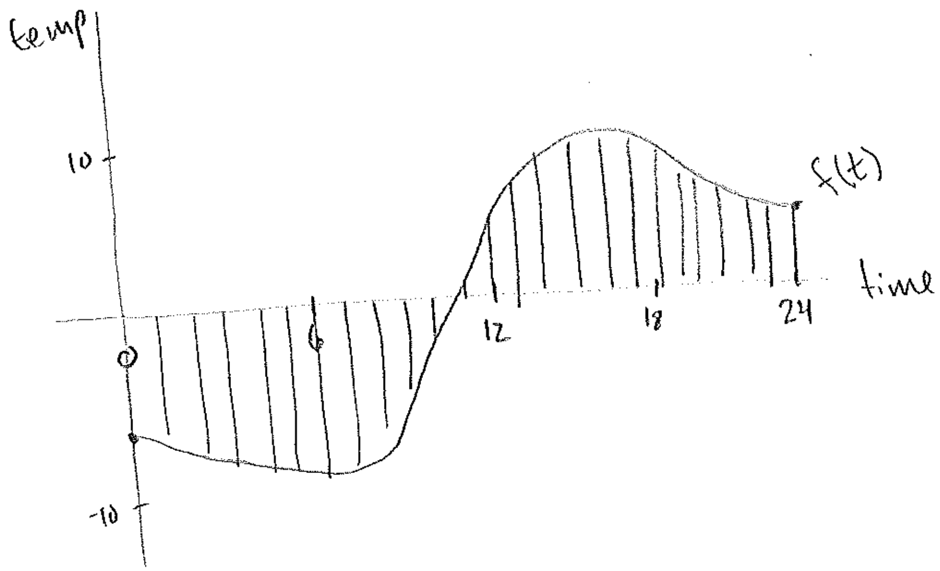
§6.5 Average Value of a Function:



Q: what is the average temperature during the day?

A: easy if only finite number of values.

Approximate the average value using $n = 24$.

$$\text{Average} \approx \frac{f(1) + f(2) + \cdots + f(24)}{24}$$

using n-subintervals $\quad \text{Average} \approx \dfrac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$

We know $\Delta x = \dfrac{b-a}{n}$ so $n = \dfrac{b-a}{\Delta x}$

$$\text{Average} \approx \frac{f(x_1) + \cdots + f(x_n)}{b-a/\Delta x} = \frac{\Delta x \left(f(x_1) + \cdots + f(x_n) \right)}{b-a}$$

$$= \frac{f(x_1)\Delta x + \cdots + f(x_n)\Delta x}{b-a} = \frac{\sum_{i=1}^{n} f(x_i)\Delta x}{b-a}$$

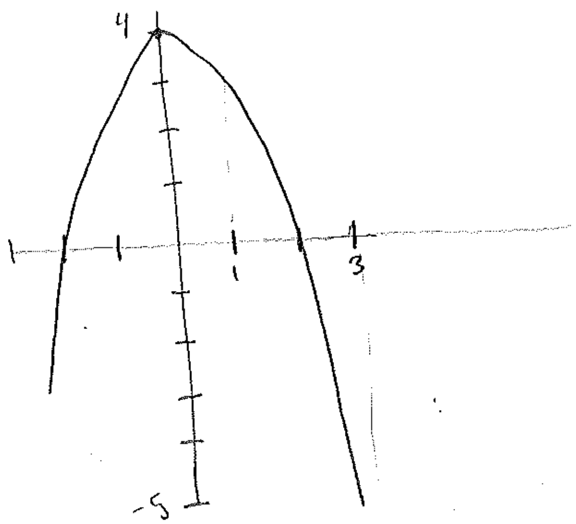
$$\text{Average} = \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

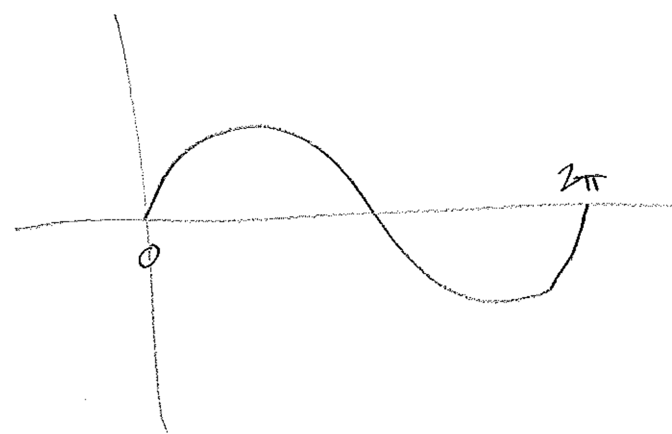
ex) find average value of $f(x) = x^2+3$ on $[0,3]$

$$\text{Average} = \frac{1}{3-0}\int_0^3 x^2+3\,dx = \frac{1}{3}\left(\frac{x^3}{3}+3x\right)\Big|_0^3 = \frac{1}{3}(9+9) = 6$$

ex) find average value of $f(x) = 4-x^2$ from $[1,3]$

$$\text{Avg} = \frac{1}{3-1}\int_1^3 4-x^2\,dx = \frac{1}{2}\left(4x-\frac{x^3}{3}\right)\Big|_1^3 = \frac{1}{2}(12-9) - \frac{1}{2}\left(4-\frac{1}{3}\right) = \frac{-1}{3}$$





ex) find average value of $f(x) = \sin x$ on $[0, 2\pi]$

$$\text{Avg} = \frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = \frac{1}{2\pi}(-\cos x)\Big|_0^{2\pi} = \frac{-1}{2\pi} - \left(\frac{-1}{2\pi}\right) = 0$$

Chapter 7: Techniques of Integration:

Basic: $\int x^2 + 3x \, dx = \frac{x^3}{3} + \frac{3}{2}x^2 + C$

More complicated: $\int x\sqrt{1-x^2}\, dx$ $\quad u = 1-x^2$, $du = -2x\,dx$ ...

§ 7.1 Integration by Parts

Start with Product Rule: $\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x)$

now "integrating" both sides

$$\int f(x)\,g'(x) + g(x)\,f'(x)\, dx = f(x)\cdot g(x)$$

$$\int f(x)\,g'(x)\,dx + \int g(x)\,f'(x)\,dx = f(x)\cdot g(x)$$

$$\int f(x)\,g'(x)\,dx = f(x)\cdot g(x) - \int g(x)\,f'(x)\,dx$$

Now let $u = f(x)$ $\quad v = g(x)$

$du = f'(x)dx$ $\quad dv = g'(x)\,dx$

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

Formula for Integration by Parts

ex] $\int x \cdot \sin x \, dx$ $\qquad u = x \qquad dv = \sin x \, dx$

$\qquad du = dx \qquad v = -\cos x$

$$= \int u \, dv = uv - \int v \, du = x(-\cos x) - \int -\cos x \, dx$$

$$= -x\cos x + \sin x + C$$

Check: $\frac{d}{dx}\left[-x \cdot \cos x + \sin x + c\right] = -\cos x - x(-\sin x) + \cos x + 0$

$$= x \cdot \sin x$$

Remark: just like u-sub is chain rule in reverse

integration by parts is product rule in reverse

note: a good choice for u is a function that becomes simpler when differentiated.

ex] $\int \ln x \, dx$ $\qquad u = \ln x \qquad dv = dx$

$\qquad du = \frac{1}{x} dx \qquad v = x$

$$= x\ln x - \int x \cdot \frac{1}{x} dx = x\ln x - \int dx = x\ln x - x + C$$