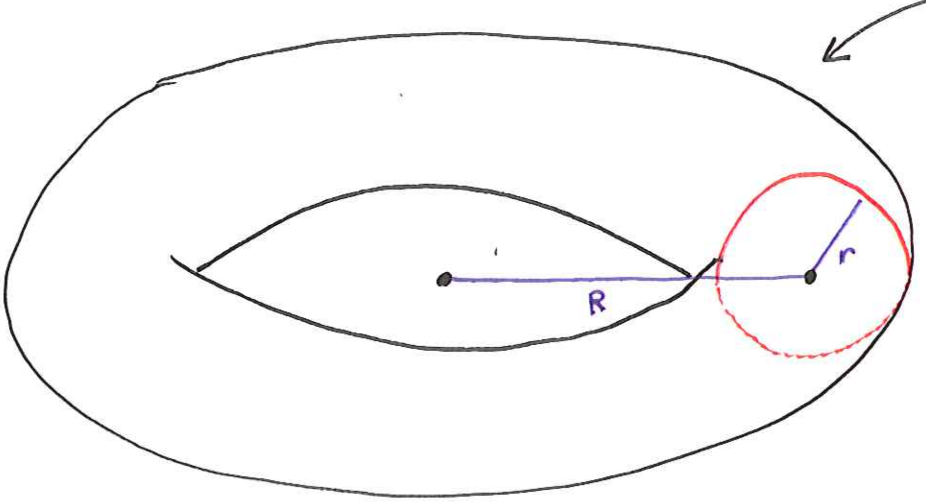
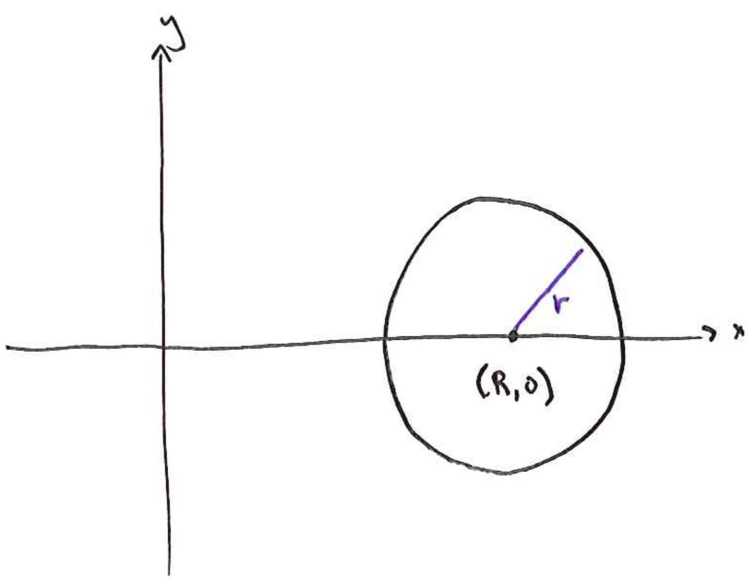
Math 2: Homework Hint Problem 6.2.61.

In this problem you are asked to compute the volume of a torus, which is the mathematical word for donut. The torus can be described by two radii:

A mathematical donut.



If you want to apply techniques from class to compute the volume, you need to recognize the torus as a solid of revolution. How can we do this?



Take the circle of radius r centered at $(R,0)$ in the plane. Rotate this about the y-axis. What solid do you get? A torus!

Because we are rotating this circle around the y-axis, to get the volume of the torus we should look at $V = \int_a^b A(y)\,dy$ ← an integral in the variable y.

What is the area function $A(y)$? We need to know something about cross-sections.

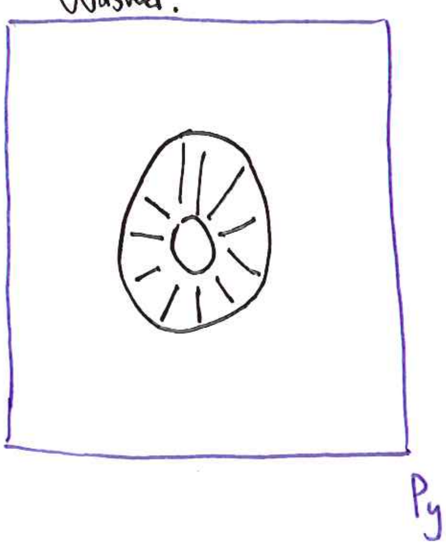
Because we are rotating about the y-axis, our cross sections are perpendicular to the y-axis.

Let P_y be the plane perpendicular to the y-axis through y.

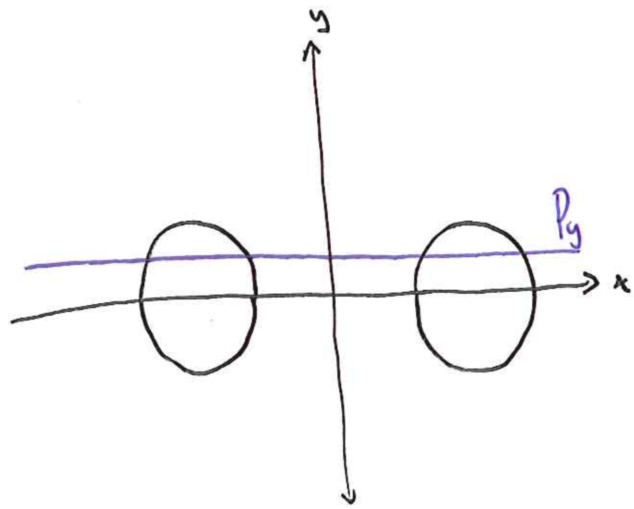
Cross section at P_y:

Top View

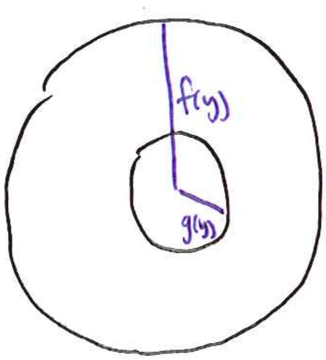
Washer.



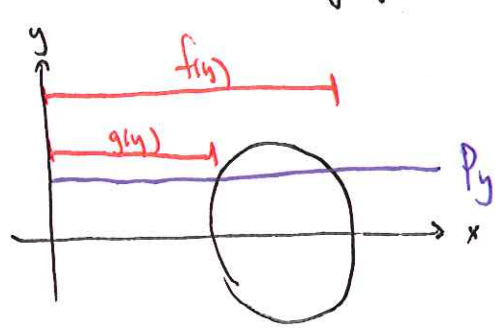
Side View



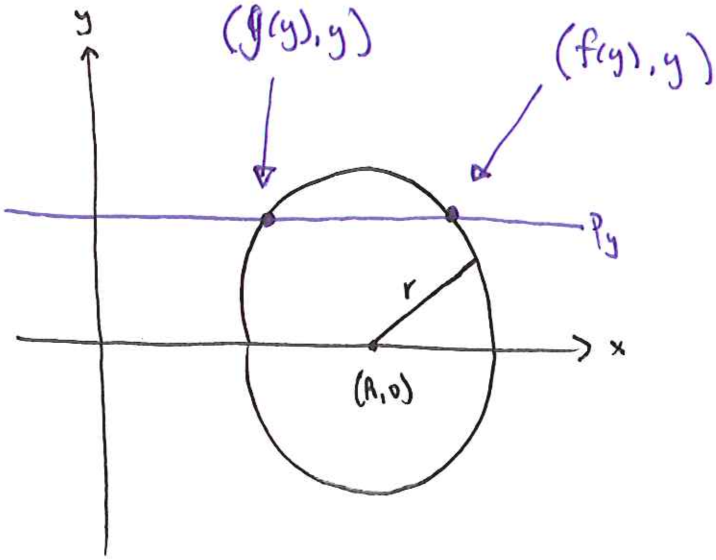
So $A(y)$ will be the area of the washer pictured above. Let $f(y)$ denote its outer radius and $g(y)$ denote its inner radius. Then



$$A(y) = \pi f(y)^2 - \pi g(y)^2.$$

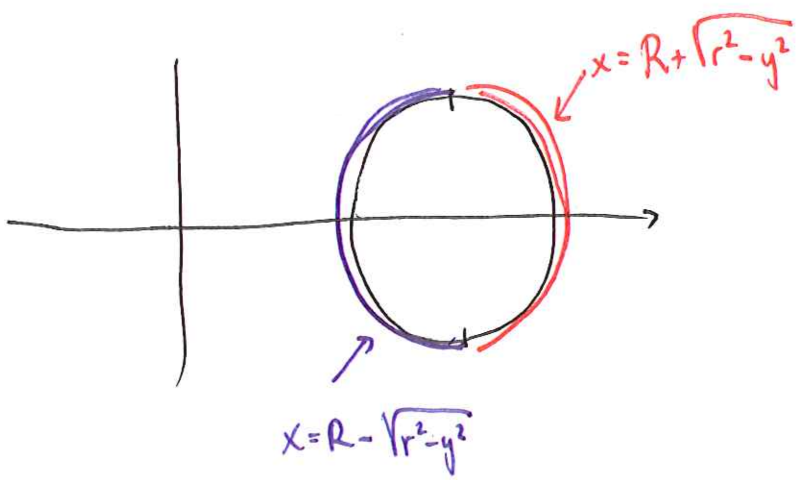


Let's find $f(y)$ and $g(y)$. In our original picture; the functions $f(y)$ and $g(y)$ appear as coordinates:



What is the equation of the above circle? It has radius r and is centered at $(R, 0)$, so it's given by $(x-R)^2 + (y-0)^2 = r^2$. We can solve for x:

$$x = R \pm \sqrt{r^2 - y^2}.$$



This gives $g(y) = R - \sqrt{r^2 - y^2}$ and $f(y) = R + \sqrt{r^2 - y^2}$.

Therefore $$A(y) = \pi\left(R + \sqrt{r^2 - y^2}\right)^2 - \pi\left(R - \sqrt{r^2 - y^2}\right)^2.$$

Expect some nice cancellation after you expand this.