Math 2 Name: ____________________

January 11, 2008

Quiz 1

Solutions

Show your work, and write clearly. No textbooks, notes, or calculators.

1. Find the following limits, if they exist:

   (a) $\lim_{x\to 0} \frac{|x|}{x}$ DNE

$$\begin{aligned} \lim_{x\to 0^+} \frac{|x|}{x} &= 1 \\ \lim_{x\to 0^-} \frac{|x|}{x} &= -1 \\ 1 &\neq -1 \end{aligned}$$

   (b) $\lim_{x\to\infty} \ln x = \infty$

   (c) $\lim_{n\to\infty} \left(\frac{81}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \frac{54}{n^2} \left(\frac{n(n+1)}{2} \right) \right) = -\frac{27}{4}$

$$\begin{aligned} \frac{81}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \frac{54}{n^2} \left(\frac{n(n+1)}{2} \right) &= \frac{81}{n^4} \left(\frac{n^2(n^2+2n+1)}{4} \right) - \frac{54}{n^2} \left(\frac{n^2+n}{2} \right) \\ &= \frac{81}{4} \left(\frac{n^4+2n^3+n^2}{n^4} \right) - \frac{54}{2} \left(\frac{n^2+n}{n^2} \right) \\ &= \frac{81}{4} \left(\frac{n^4}{n^4} + \frac{2n^3}{n^4} + \frac{n^2}{n^4} \right) - \frac{54}{2} \left(\frac{n^2}{n^2} + \frac{n}{n^2} \right) \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{81}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \frac{54}{n^2} \left(\frac{n(n+1)}{2} \right) \right) &= \lim_{n\to\infty} \frac{81}{4} \left(\frac{n^4}{n^4} + \frac{2n^3}{n^4} + \frac{n^2}{n^4} \right) - \frac{54}{2} \left(\frac{n^2}{n^2} + \frac{n}{n^2} \right) \\ &= \frac{81}{4} \left(\lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \right) - \frac{54}{2} \left(\lim_{n\to\infty} \left(1 + \frac{1}{n} \right) \right) \\ &= \frac{81}{4} (1+0+0) - \frac{54}{2} (1+0) \\ &= -\frac{-27}{4} \end{aligned}$$

2. Find the derivative of $f(\theta) = e^{\sin(\theta^2)}$.
This uses the chain rule. $f(\theta) = g(h(i(\theta)))$ where $g(x) = e^x$, $h(x) = \sin x$, and $i(x) = x^2$.

$$\begin{aligned} g'(x) &= e^x \\ h'(x) &= \cos x \\ i'(x) &= 2x \\ f'(\theta) &= g'(h(i(\theta))h'(i(\theta))i'(\theta) \\ &= (e^{\sin(\theta^2)})(\cos(\theta^2))(2\theta) \\ &= 2\theta\cos(\theta^2)e^{\sin(\theta^2)} \end{aligned}$$

3. Find the derivative of $y = \cos(2x)$ in two different ways.
   (a) First, use the chain rule.
   (b) Second, use the formula for cosine of a double angle: $\cos(2x) = \cos^2 x - \sin^2 x$.
   (c) Check that your answers agree by using the formula for sine of a double angle: $\sin 2x = 2\sin x\cos x$.

   (a) $y' = -2\sin(2x)$

   (b)

$$\begin{aligned} y &= \cos(2x) \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

$$\begin{aligned} y' &= 2\cos x(-\sin x) - 2\sin x(\cos x) \\ &= -4\sin x\cos x \end{aligned}$$

   (c)

$$\begin{aligned} \text{(a)} &= -2\sin(2x) \\ &= -2\,(2\sin x\cos x) \\ &= -4\sin x\cos x = \text{(b)} \qquad \checkmark \end{aligned}$$

4. Suppose you leave Hanover right now by car, to get as far away from this quiz as possible. Your distance traveled (in miles) is given by $d(t) = 30t^2 + 20t$, where t is the time, in hours. The speed limit is 65 miles per hour, but the cops won't pull you over until youre going 80 miles per hour.
   (a) Assuming that the area is heavily patrolled and that you'll be pulled over as soon as you hit 80 mph, how long will it be until you get pulled over?

(b) How many miles will you have traveled by then?

(c) Suppose now that you don't stop for the cops, and continue to flee the quiz as planned. How far away from Hanover will you be one hour after you see the flashing lights in your mirror?

(d) What will your speed be at this time?

$$\begin{aligned} d(t) &= 30t^2 + 20t \\ v(t) = d'(t) &= 60t + 20 \end{aligned}$$

(a) We want to know what t is when v is 80.

$$\begin{aligned} v(t) &= 80 \\ 60t + 20 &= 80 \\ 60t &= 60 \\ t &= 1 \end{aligned}$$

You will be pulled over after 1 hour of driving.

(b) We want to know what d is when t is 1 hour.

$$\begin{aligned} d(1) &= 30(1)^2 + 20(1) \\ &= 30 + 20 \\ &= 50 \end{aligned}$$

So after one hour you have traveled 50 miles.

(c) An hour after the flashing lights will be 2 hours total, so we want to know d when t is 2.

$$\begin{aligned} d(2) &= 30(2)^2 + 20(2) \\ &= 30(4) + 40 \\ &= 120 + 40 \\ &= 160 \end{aligned}$$

So an hour after the flashing lights you will have travel 160 miles.

(d) Now, we want to know v when t is 2.

$$\begin{aligned} v(2) &= 60(2) + 20 \\ &= 120 + 20 \\ &= 140 \end{aligned}$$

An hour after the flashing lights, you will be traveling 140 miles per hour, that's quite fast.