Math 2 January 28, 2008

Practice Exam 1

This is intended to closely resemble the actual exam in terms of difficulty of questions, however it is somewhere between 1.5 and 2 exams in length. Doing these problems is most likely not enough to prepare yourself for the exam. Studying your homework, the quizzes, and the Chapter 5 Review in the textbook first, and then using this exam as a way to test yourself is a good idea. You should not need a calculator for these problems, and they will not be allowed during the exam.

1. TRUE or FALSE. Mark each statement as TRUE or FALSE. These will not be given partial credit.

   (a) If f is continuous on $[a, b]$, then $\int_a^b x f(x)\,dx = x \int_a^b f(x)\,dx$. FALSE

   - x is not a constant
   - left side is a number ; right side is a function of x

   (b) If f and g are continuous on $[a, b]$, then $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

   TRUE

   Section 5.2 gives other properties of integrals too.

   (c) If h is a continuous function, $\int_{-3}^{5} h(x)\,dx = 12$, and $\int_2^5 h(x)\,dx = -4$, then we know $\int_{-3}^{2} h(x)\,dx = 16$. TRUE

   $$\underbrace{\int_{-3}^{5} h(x)dx}_{12} = \underbrace{\int_{-3}^{2} h(x)dx}_{?} + \underbrace{\int_{2}^{5} h(x)dx}_{-4} \qquad ? = 16$$

   (d) If f is a continuous function, then TRUE

   that "x" shouldn't be there

   $$\frac{d}{dx}\int_{-x}^{x} f(t)\,dt = \frac{d}{dx}\left(\int_0^x f(t)\,dt\cancel{x} - \int_0^{-x} f(t)\,dt\right) = f(x) + f(-x)$$

   $$\frac{d}{dx}\int_{-x}^{x} f(t)dt = \frac{d}{dx}\left(\int_{-x}^{0} f(t)dt + \int_0^x f(t)dt\right)$$

   $$= \frac{d}{dx}\left(-\int_0^{-x} f(t)dt\right) + \frac{d}{dx}\left(\int_0^x f(t)dt\right)$$

   $$= [f(-x)]\underbrace{(-1)}_{=(-x)'} + [f(x)] = f(x) + f(-x)$$

2. **Look on page 392 for the answers.**

3. Compute the following integral in three ways:

$$\int_{-2}^{2} 2x - 1\,dx$$

(a) Use the definition of a definite integral as the limit of a Riemann Sum.

$a = -2$

$b = 2$

$\Delta x = \frac{b-a}{n} = \frac{4}{n}$

$x_i = a + i\Delta x$

$= -2 + i\left(\frac{4}{n}\right)$

$= -2 + \frac{4i}{n}$

$f(x_i) = 2\left(-2 + \frac{4i}{n}\right) - 1$

$= -4 + \frac{8i}{n} - 1$

$= \frac{8i}{n} - 5$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\int_{-2}^{2} 2x - 1\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n} - 5\right)\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i}{n^2} - \frac{20}{n}$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{32i}{n^2} + \sum_{i=1}^{n} -\frac{20}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{32}{n^2} \sum_{i=1}^{n} i - \frac{20}{n} \sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{32}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{20}{n}(n)\right)$$

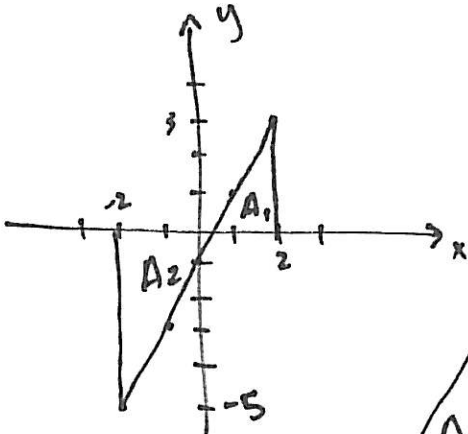
$$= \lim_{n\to\infty} \left(\frac{32}{2} \cdot \frac{n+1}{n} - 20\right)$$

$$= \frac{32}{2} \cdot (1) - 20 = 16 - 20 = -4$$

$$\boxed{\int_{-2}^{2} 2x - 1\,dx = -4}$$ ✓

(b) Use geometry and the fact that a definite integral represents a net area.

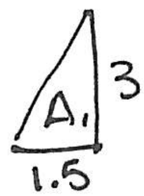
$f(x) = 2x - 1$



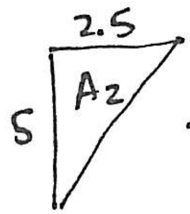
$$\int_{-2}^{2} 2x - 1 \, dx = A_1 - A_2$$

$$= \frac{9}{4} - \frac{25}{4} = -\frac{16}{4} = -4$$

$$\boxed{\int_{-2}^{2} 2x - 1 \, dx = -4}$$ ✓



$\rightarrow A_1 = \frac{1}{2}(3)\left(\frac{3}{2}\right) = \frac{9}{4}$



$\rightarrow A_2 = \frac{1}{2}(5)\left(\frac{5}{2}\right) = \frac{25}{4}$

(c) Use the second part of the Fundamental Theorem of Calculus.

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{where } F'(x) = f(x)$$

$$f(x) = 2x - 1 \longrightarrow F(x) = x^2 - x$$

$$\int_{-2}^{2} 2x - 1 \, dx = x^2 - x \Big|_{-2}^{2} = \left[(2)^2 - (2)\right] - \left[(-2)^2 - (-2)\right]$$

$$= \left[4 - 2\right] - \left[4 - (-2)\right]$$

$$= 2 - 6 = -4$$

$$\boxed{\int_{-2}^{2} 2x - 1 \, dx = -4}$$ ✓

4. Find the derivative of the function g(x) in two ways:

$$g(x) = \int_0^{3\sqrt{x}} t^2 + 1\, dt$$

(a) Use Part 1 of the Fundamental Theorem to find $g'(x)$

$g(x) = \int_a^x f(t)dt \longrightarrow g'(x) = f(x)$

$f(t) = t^2+1 \qquad g(x) = h(3\sqrt{x})$ where $h(x) = \int_0^x t^2+1dt$

$h'(x) = x^2+1$

So $g'(x) = h'(3\sqrt{x})(3\sqrt{x})'$

$= ((3\sqrt{x})^2+1)(\frac{3}{2}x^{-1/2})$

$= (9x+1)\left(\frac{3}{2\sqrt{x}}\right) = \frac{27x}{2\sqrt{x}} + \frac{3}{2\sqrt{x}} = \frac{27}{2}x^{1/2} + \frac{3}{2}x^{-1/2}$

(b) Use Part 2 of the Fundamental Theorem to evaluate the definite integral,

$$\int_0^{3\sqrt{x}} t^2 + 1\, dt$$

, then take its derivative.

$g(x) = \int_0^{3\sqrt{x}} t^2+1\, dt = \frac{t^3}{3} + t \Big|_0^{3\sqrt{x}} = \left[\frac{(3\sqrt{x})^3}{3} + (3\sqrt{x})\right] - \left[\cancel{\frac{0^3}{3}+0}\right]$ 0

$= \frac{27x^{3/2}}{3} + 3x^{1/2} = 9x^{3/2} + 3x^{1/2}$

$g'(x) = 9\left(\frac{3}{2}\right)x^{1/2} + 3\left(\frac{1}{2}\right)x^{-1/2}$

$= \frac{27}{2}x^{1/2} + \frac{3}{2}x^{-1/2}$

5. Compute the following integrals.

(a) $\int_1^4 \sqrt{x^3}\,dx$

$$= \int_1^4 (x^3)^{1/2} dx = \int_1^4 x^{3/2} dx$$

$$= \frac{x^{5/2}}{5/2} \Big|_1^4 = \frac{2}{5}\left[(4)^{5/2} - (1)^{5/2}\right]$$

$$= \frac{2}{5}(2^5 - 1) = \frac{2}{5}(32-1) = \frac{62}{5}$$

(b) $\int 2^x\,dx$

$$= \frac{2^x}{\ln 2} + C$$

(c) $\int_1^5 \frac{3}{x}\,dx$

$$= 3\int_1^5 \frac{1}{x} dx = 3\left[\ln x\right]_1^5$$

$$= 3\left[\ln 5 - \ln 1\right]$$

$$= 3\ln 5$$

(d) $\int 7t\cos(t^2)\,dt$ $= \int \cos(u)\left(\frac{7}{2}du\right)$

$u = t^2$ $= \frac{7}{2}\int \cos u\,du$

$du = 2t\,dt$ $= \frac{7}{2}(\sin u) + C$

$\frac{7}{2}du = 7t\,dt$

$= \frac{7}{2}\sin(t^2) + C$

(e) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2\theta\,d\theta$ $= \tan\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \tan\left(\frac{\pi}{3}\right) - \tan\left(-\frac{\pi}{4}\right)$

$= \dfrac{\sin \pi/3}{\cos \pi/3} - \dfrac{\sin(-\pi/4)}{\cos(-\pi/4)} = \dfrac{\sqrt{3}/2}{1/2} - \dfrac{-\sqrt{2}/2}{\sqrt{2}/2}$

$= \sqrt{3} - (-1) = \sqrt{3} + 1$

(f) $\int x^3\sqrt{1+x^4}\,dx$ $= \int \sqrt{u}\left(\frac{du}{4}\right) = \frac{1}{4}\int u^{1/2}\,du$

$u = 1 + x^4$

$= \frac{1}{4}\left[\frac{u^{3/2}}{3/2}\right] + C$

$du = 4x^3\,dx$

$\frac{du}{4} = x^3\,dx$ $= \frac{1}{4}\cdot\frac{2}{3}(1+x^4)^{3/2} + C$

$= \dfrac{(1+x^4)^{3/2}}{6} + C$

(g) $\int_1^2 \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = \int_1^{\sqrt{2}} e^u (2du) = 2\int_1^{\sqrt{2}} e^u du$

$u = \sqrt{y}$

$du = \frac{1}{2\sqrt{y}}dy$

$2du = \frac{dy}{\sqrt{y}}$

$y=1 \longrightarrow u=\sqrt{1}=1$

$y=2 \longrightarrow u=\sqrt{2}$

$= 2\left[e^u\right]_1^{\sqrt{2}}$

$= 2\left(e^{\sqrt{2}} - e^1\right)$

$= 2e^{\sqrt{2}} - 2e$

(h) $\int \cos\theta e^{\sin\theta}\,d\theta = \int e^u du = e^u + C$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= e^{\sin\theta} + C$

6. Using the terms we've learned so far, explain in a short paragraph the relationship between the following ideas as they apply to the motion of a particle moving along a straight line:

- DISTANCE TRAVELED, or TOTAL DISTANCE
- ACCELERATION
- VELOCITY
- SPEED
- DISPLACEMENT, or NET DISTANCE

If the velocity of the particle is given by $v(t)$, then the speed is the absolute value of velocity, so $|v(t)|$ is the speed.

Acceleration, is the derivative of velocity, so $a(t) = v'(t)$.

The total distance traveled is the integral of the speed. So, $\int_0^T |v(t)|dt$ is the total distance traveled from time 0 to time T.

The total displacement, or net distance, is given by the integral of velocity: $\int_0^T v(t)dt$ gives the displacement over the time interval $[0,T]$.

7. A ball is thrown straight up, from the edge of a cliff 40 feet high, at 25 ft/sec, and the acceleration of gravity is 32 ft/sec^2 downwards, i.e. -32 ft/sec^2. Find

(a) the velocity function,

$a(t) = -32$ & $v(t) = \int a(t)\,dt$

$v(t) = -32t + C$ $\quad v(0) = 25 \longrightarrow v(t) = -32t +$

(b) the speed function,

speed $= |v(t)| = |-32t+25|$

(c) total distance function,

total distance $= \int |v(t)|\,dt = \int |-32t+25|\,dt$

(d) displacement, or net distance, function,

displacement over $[0,T] = \int_0^T v(t)\,dt$

$= \int_0^T -32t+25\,dt = -16t^2+25t\Big|_0^T = -16T^2+25T$

(e) how long the ball is in the air,

when is displacement is -40 ft (cliff height)

$-16T^2+25T=-40$ $\quad T = \dfrac{-25 \pm \sqrt{(25)^2-4(-16)(40)}}{2(-16)} \rightarrow T = 2.54s$

$-16T^2+25T+40=0$

(f) when it reaches its max,

when velocity $=0$

$-32t+25=0 \Rightarrow t = \frac{25}{32}$ s.

(g) how high this is, (the max height),

position $= \int v(t)\,dt = -16t^2+25t+C$ $\quad$ at $t=0$ pos. $=40$ft

position at $\frac{25}{32}$s $= -16\left(\frac{25}{32}\right)^2 + 25\left(\frac{25}{32}\right) + 40 = 49.76$ft

(h) and how fast it's going when it hits the ground.

$v(t)$ when pos $= 0$ ft ($t = 2.54$s by part (e))

$v(2.54s) = -32(2.54)+25 = -56.28$ ft/s

or 56.28 ft/s downward.