1. SHORT ANSWER You do not need to provide reasons for you answers.

(a) [4 points] CIRCLE ONE If the DERIVATIVES or ANTIDERIVATIVES of two functions are equal, then the functions are equal.

(b) [4 points] FILL IN THE BLANK If f is a continuous function on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(t)\,dt\right) = \underline{\ 0\ }.$$

(c) [4 points] FILL IN THE BLANK If $f(x) \leqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \underline{\ \leq\ } \int_a^b g(x)\,dx.$$

(d) [4 points] TRUE OR FALSE $\underline{\ F\ }$ It is possible, and legal, to rewrite

$$\int \frac{x^2+4}{x^2(x-4)}\,dx \quad \text{as} \quad \int \frac{A}{x^2} + \frac{B}{x-4}\,dx.$$

2. Paul has been working all winter on an amazing snow-ball-gun that shoots snowballs at a speed of 24 ft/s. Last weekend he climbed up Mt Cardigan, again, to test out his shooter from the top of the 40 feet tall observation tower. Once he reaches the top of the mountain he fills a bucket up to the top full of snow, making it weigh a whopping 20 lbs. Once he gets to the top, he hangs the bucket of snow on a hook from the ceiling, and as he lets go of the bucket it lowers 1 foot, because the hook was hanging from a spring. He then loads his gun and shoots a snowball straight up into the air.

   (a) [12 points] How much work would be needed to stretch the same spring that the bucket was hanging on 1 more foot, for a total of two feet of stretching?

$F(x) = kx$ $\qquad$ $F(x) = 20x$

$20\text{lb} = k\,(1\text{ft})$

$k = 20$

$W = \int_1^2 F(x)dx = \int_1^2 20x\,dx = 10x^2\Big|_1^2 = 10(4-1) = \boxed{30\text{ ft-lbs}}$

(b) [8 points] Give the equation for the velocity, $v(t)$, of the snow ball after it is shot out of the gun. Remember the acceleration of gravity is -32 ft/s^2.

$$v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C$$

$$v(0) = 24$$
$$24 = -32(0) + C$$
$$C = 24$$

$$\boxed{v(t) = -32t + 24}$$

(c) [8 points] Use your equations from part (b) to calculate the total distance traveled before it hits the ground again. (The snowball stays in the air for 2.5 seconds.)

$$\text{DIST} = \int_0^{2.5} |v(t)|\,dt$$

$$= \int_0^{3/4} -32t+24\,dt + \int_{3/4}^{5/2} 32t-24\,dt$$

$$= -16t^2+24t\Big|_0^{3/4} + 16t^2-24t\Big|_{3/4}^{5/2}$$

$$= \left(-16\left(\tfrac{3}{4}\right)^2+24\left(\tfrac{3}{4}\right)\right)-(0) + \left(16\left(\tfrac{5}{2}\right)^2-24\left(\tfrac{5}{2}\right)\right)-\left(16\left(\tfrac{3}{4}\right)^2-24\left(\tfrac{3}{4}\right)\right)$$

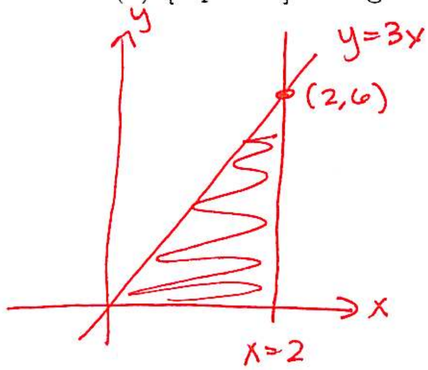
$$= -9+18+100-60-9+18$$

$$= 58\text{ ft}$$

$$-32t+24=0$$
$$t = \frac{24}{32} = \frac{3}{4}$$
$$v(1) = -8 < 0$$
$$v(\tfrac{1}{2}) = 8 > 0$$

3. Find the area of the region R bounded by the lines $y = 3x$, $x = 2$ and $y = 0$ in three ways:

   (a) [4 points] Use geometric formulas.



$A = \frac{1}{2}bh$

$= \frac{1}{2}(2)(6)$

$= 6$

   (b) [8 points] Set up, and evaluate and integral representing the area between two curves.

$A = \int_0^2 \text{top} - \text{bottom}$

$= \int_0^2 (3x) - (0)\,dx$

$= \int_0^2 3x\,dx$

$= \frac{3}{2}x^2 \Big|_0^2 = \frac{3}{2}(4-0) = 6$

(c) [8 points] Use the definition of an integral to rewrite you integral above as a limit of a Riemann Sum, and evaluate.

$a=0$

$b=2$

$f(x)=3x$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + i\frac{2}{n} = \frac{2i}{n}$

$f(x_i) = 3\left(\frac{2i}{n}\right) = \frac{6i}{n}$

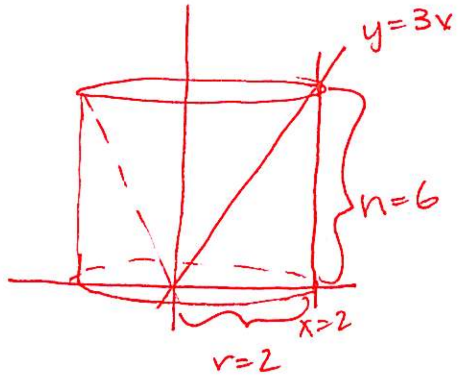
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{12i}{n^2}$$

$$= \lim_{n\to\infty} \frac{12}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{12}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{12}{2}(1) = 6$$

4. Find the volume of the solid formed when the region above, problem 3, is rotated about the line y-axis in three ways:

   (a) [4 points] Use geometric formulas for the volume of typical solids.



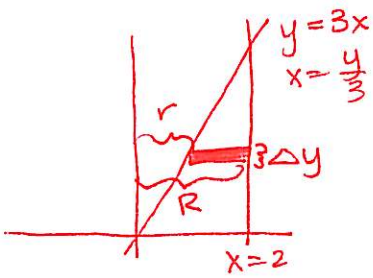
$V = \text{cylinder} - \text{cone}$

$= \pi r^2 h - \frac{1}{3}\pi r^2 h$

$= \frac{2}{3}\pi r^2 h$

$= \frac{2}{\cancel{3}}\pi (2)^2 (\cancel{6}^{\,2}) = 16\pi$

   (b) [12 points] Use integration with the washer method.



$V_{washer} = \pi (R^2 - r^2)\Delta y$

$= \pi\left((2)^2 - \left(\frac{y}{3}\right)^2\right)\Delta y$

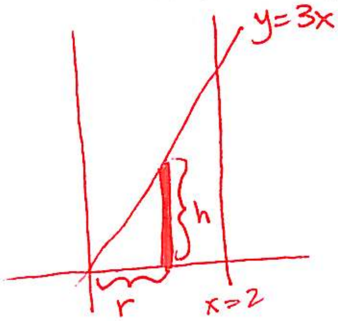
$= \pi\left(4 - \frac{y^2}{9}\right)\Delta y$

$V = \int_0^6 \pi\left(4 - \frac{y^2}{9}\right) dy = \pi\left[4y - \frac{y^3}{27}\right]_0^6$

$= \pi\left[4(6) - \frac{(6)^3}{27} - 0\right]$

$= \pi\left(24 - \frac{\cancel{6}^{\,2}\cdot\cancel{6}^{\,2}\cdot\cancel{6}^{\,2}}{\cancel{3}\cdot\cancel{3}\cdot\cancel{3}}\right) = (24-8)\pi = 16\pi$

(c) [12 points] Use integration with the method of cylindrical shells.



$V_{shell} = 2\pi r h \, \Delta x$

$= 2\pi (x)(3x) \, \Delta x$

$= 6\pi x^2 \, \Delta x$

$V = \int_0^2 6\pi x^2 dx = 2\pi x^3 \Big|_0^2 = 2\pi (2)^3 = 16\pi$

5. Evaluate the following integrals using any method you'd like.

(a) [16 points]

$$\int_0^{\frac{\pi}{2}} \sin^{10}\theta \cos^3\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{10}\theta \cos^2\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{10}\theta (1-\sin^2\theta) \cos\theta \, d\theta$$

$u = \sin\theta \quad du = \cos\theta \, d\theta$

$\theta = 0 \rightarrow u = 0 \quad \theta = \pi/2 \rightarrow u = 1$

$$= \int_0^1 u^{10}(1-u^2) \, du$$

$$= \int_0^1 u^{10} - u^{12} \, du$$

$$= \frac{u^{11}}{11} - \frac{u^{13}}{13} \Big|_0^1 = \frac{1}{11} - \frac{1}{13} = \frac{2}{143}$$

(b) [16 points]

$$\int_e^{e^3} \frac{\ln(\ln x)}{x \ln x}\, dx$$

$u = \ln(\ln x)$

$du = \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$x = e \rightarrow u = \ln(\ln(e)) = \ln(1) = 0$

$x = e^3 \rightarrow u = \ln(\ln(e^3)) = \ln(3)$

$$\int_e^{e^3} \frac{\ln(\ln x)}{x \ln x} dx = \int_0^{\ln(3)} u\, du$$

$$= \frac{u^2}{2}\Big|_0^{\ln(3)} = \frac{(\ln 3)^2}{2} - 0$$

$$= \frac{1}{2}(\ln 3)^2$$

(c) [16 points]

$$\int \frac{1}{x^2 + x - 6}\, dx$$

$$x^2+x-6 = (x+3)(x-2)$$

$$\frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x-2)+B(x+3)}{(x+3)(x-2)} = \frac{(A+B)x+(-2A+3B)}{(x+3)(x-2)}$$

$$(A+B)x + (-2A+3B) = 1$$

$$A+B=0 \longrightarrow A=-B \qquad A=-\tfrac{1}{5}$$

$$-2A+3B=1 \longrightarrow -2(-B)+3B=1 \longrightarrow 5B=1 \longrightarrow B=\tfrac{1}{5}$$

$$\int \frac{1}{x^2+x-6}\,dx = \int \frac{-\frac{1}{5}}{x+3} + \frac{\frac{1}{5}}{x-2}\,dx$$

$$= -\frac{1}{5}\ln|x+3| + \frac{1}{5}\ln|x-2| + C$$

$$= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C$$

(d) [16 points]

$$\int \frac{\sqrt{x^2-1}}{x}\,dx$$

$x = \sec\theta$ $\qquad$ $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

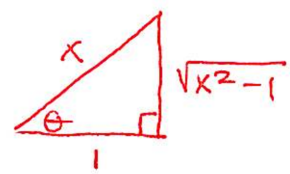
$dx = \sec\theta \tan\theta\, d\theta$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\cancel{\sec\theta}} \cancel{\sec\theta} \tan\theta\, d\theta$$

$\sec\theta = x$

$\cos\theta = \frac{1}{x}$

$$= \int \tan^2\theta \, d\theta$$



$$= \int \sec^2\theta - 1\, d\theta$$

$$= \tan\theta - \theta + C$$

$$= \sqrt{x^2-1} - \cos^{-1}\left(\frac{1}{x}\right) + C$$

6. (a) [12 points] Find the average value of the function $f(x) = \ln x$ on the interval $[e^2, e^3]$.

$$f_{ave} = \frac{1}{e^3-e^2}\int_{e^2}^{e^3} \ln x \, dx$$

$u = \ln x \quad du = \frac{1}{x}dx$

$dv = dx \quad v = x$

$$= \frac{1}{e^3-e^2}\left[x\ln x \Big|_{e^2}^{e^3} - \int_{e^2}^{e^3} \frac{x}{x}\,dx\right]$$

$$= \frac{1}{e^3-e^2}\left[e^3\ln e^3 - e^2\ln e^2 - x\Big|_{e^2}^{e^3}\right]$$

$$= \frac{1}{e^3-e^2}\left[3e^3 - 2e^2 - e^3 + e^2\right]$$

$$= \frac{2e^3-e^2}{e^3-e^2} = \frac{e^2(2e-1)}{e^2(e-1)} = \frac{2e-1}{e-1}$$

(b) [4 points] Is this average value attained somewhere in the interval? If so, how many times and where?

YES!

$$f(c) = f_{ave}$$

$$\ln c = \frac{2e-1}{e-1}$$

$$c = e^{\frac{2e-1}{e-1}}$$

7. [20 points] Prove **one** of the following area formulas from geometry:

(a) The area of a circle with radius r is $A = \pi r^2$.

(b) The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

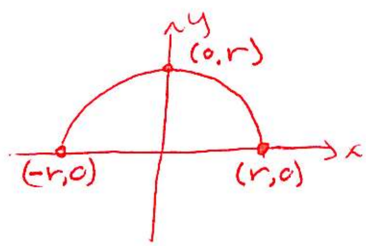
(c) The volume of a cone with height h and radius r is $V = \frac{1}{3}\pi r^2 h$.

a) $x^2 + y^2 = r^2$

$y = \pm\sqrt{r^2 - x^2}$

$y = \sqrt{r^2 - x}$

top half



$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

$x = r\sin\theta \qquad dx = r\cos\theta\, d\theta$

$x = -r \rightarrow \theta = \sin^{-1}(-1) = -\frac{\pi}{2}$

$x = r \rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$

$\sqrt{r^2 - x^2} = \sqrt{r^2(1-\sin^2\theta)} = r\cos\theta$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (r\cos\theta)(r\cos\theta)\, d\theta$$

$$= 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos 2\theta\, d\theta$$

$u = 2\theta$

$du = 2\, d\theta$

$\theta = -\frac{\pi}{2} \rightarrow u = -\pi$

$\theta = \frac{\pi}{2} \rightarrow u = \pi$

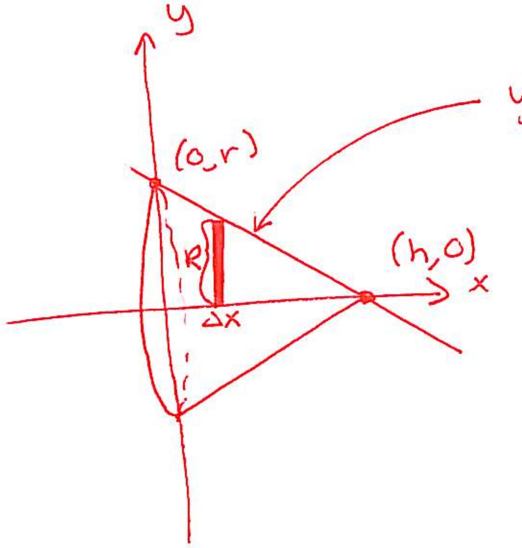
$$= r^2 \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos u)\, du$$

$$= \frac{r^2}{2}\Big[u + \sin u\Big]_{-\pi}^{\pi}$$

$$= \frac{r^2}{2}\left[\pi + \cancel{\sin\pi}^{\,0} - (-\pi + \cancel{\sin(-\pi)}^{\,0})\right]$$

$$= \frac{2\pi r^2}{2} = \pi r^2$$

b)



$y = mx + b$

$m = \frac{-r}{h} \quad b = r$

$y = -\frac{r}{h}x + r$

$V_{disk} = \pi R^2 \Delta x$

$= \pi\left(-\frac{r}{h}x + r\right)^2 \Delta x$

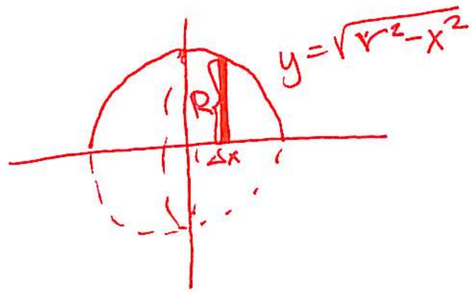
$= \pi r^2\left(\frac{x^2}{h^2} - \frac{2x}{h} + 1\right)\Delta x$

$$V = \int_0^h \pi r^2\left(\frac{x^2}{h^2} - \frac{2x}{h} + 1\right)dx$$

$$= \pi r^2\left[\frac{x^3}{3h^2} - \frac{x^2}{h} + x\right]_0^h$$

$$= \pi r^2\left[\frac{h^3}{3h^2} - \frac{h^2}{h} + h\right] = \pi r^2\left(\frac{1}{3}h - h + h\right) = \frac{1}{3}\pi r^2 h$$

c)



$V_{washer} = \pi R^2 \Delta x$

$= \pi\left(\sqrt{r^2 - x^2}\right)^2 \Delta x$

$= \pi(r^2 - x^2)\Delta x$

$$V = \int_{-r}^{r} \pi(r^2 - x^2)dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^2 \cdot r - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right]$$

$$= \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right)$$

$$= \frac{4}{3}\pi r^3$$

8. [8 points] What is your favorite math fun fact?

$$e^{i\pi} + 1 = 0$$

9. BONUS (4 points each):

   (a) Evaluate $\int \arctan x \, dx$.

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2} dx \qquad v = x$

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} dx$$

$w = 1 + x^2$

$dw = 2x dx$

$\frac{dw}{2} = x dx$

$$= x \arctan x - \frac{1}{2} \int \frac{1}{w} dw$$

$$= x \arctan x - \frac{1}{2} \ln |w| + C$$

$$= x \arctan x - \frac{\ln |1+x^2|}{2} + C$$

(b) Evaluate $\int \frac{e^x}{\sqrt{4+e^{-2x}}}\, dx$.

$$u = e^{-x} \qquad du = -e^{-x}dx = -u\,dx \longrightarrow dx = -\frac{du}{u}$$

$$e^{-2x} = (e^{-x})^2 = u^2 \qquad e^x = (e^{-x})^{-1} = u^{-1}$$

$$\int \frac{e^x}{\sqrt{4+e^{-2x}}}\,dx = \int \frac{u^{-1}}{\sqrt{4+u^2}}\left(-\frac{du}{u}\right) = -\int \frac{1}{u^2\sqrt{4+u^2}}\,du$$

$$u = 2\tan\theta \qquad \sqrt{4+u^2} = \sqrt{4(1+\tan^2\theta)} = 2\sec\theta$$

$$du = 2\sec^2\theta\, d\theta$$

$$= -\int \frac{1}{(2\tan\theta)^2(\cancel{2}\cancel{\sec\theta})}\cancel{2}\sec^2\theta\, d\theta$$

$$= -\int \frac{\sec\theta}{4\tan^2\theta}\,d\theta = -\frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= -\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \qquad \begin{aligned} v &= \sin\theta \\ dv &= \cos\theta\, d\theta \end{aligned}$$

$$= -\frac{1}{4}\int \frac{1}{v^2}\,dv = -\frac{1}{4}\left(-v^{-1}\right) + C = \frac{1}{4v} + C = \frac{1}{4\sin\theta} + C$$

$$= \frac{\sqrt{u^2+4}}{4u} + C = \frac{\sqrt{e^{-2x}+4}}{4e^{-x}} + C = \frac{1}{4}e^x\sqrt{e^{-2x}+4} + C$$