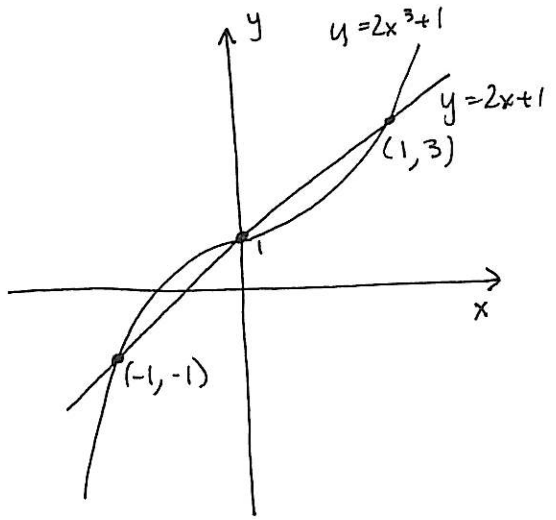
1. [20 points] Find the area of the region in the plane bounded by the graphs of $y = 2x + 1$ and $y = 2x^3 + 1$.
(Hint: This region has two parts.)



$$2x^3+1 = 2x+1$$
$$2x^3-2x=0$$
$$2x(x^2-1)=0$$
$$2x(x-1)(x+1)=0$$
$$x=0,\pm 1$$

$$A = \int_{-1}^{0} (2x^3+1)-(2x+1)\,dx + \int_0^1 (2x+1)-(2x^3+1)\,dx$$

$$= \int_{-1}^{0} 2x^3-2x\,dx + \int_0^1 2x-2x^3\,dx$$

$$= \left[\frac{2x^4}{4} - x^2\right]_{-1}^{0} + \left[x^2 - \frac{2x^4}{4}\right]_0^1$$

$$= \left[(0-0)-\left(\frac{2(-1)^4}{4}-(-1)^2\right)\right] + \left[\left(1^2-\frac{2(1)^2}{4}\right)-(0-0)\right]$$

$$= -\left(\frac{1}{2}-1\right)+\left(1-\frac{1}{2}\right) = \frac{1}{2}+\frac{1}{2} = \boxed{1}$$

2. (a) [10 points] Find the average value of $f(x) = \frac{1}{x}$ on the interval $[2, 6]$.

$$f_{ave} = \frac{1}{6-2} \int_2^6 \frac{1}{x}\, dx$$

$$= \frac{1}{4} \ln x \Big|_2^6$$

$$= \frac{1}{4}(\ln 6 - \ln 2)$$

$$= \frac{1}{4} \ln\left(\frac{6}{2}\right)$$

$$= \frac{\ln 3}{4}$$

(b) [10 points] For what number(s) c in this interval is the average value you just found actually attained, i.e. $f(c) = f_{\text{ave}}$?

$$f(c) = f_{ave}$$

$$\frac{1}{c} = \frac{\ln 3}{4}$$

$$4 = c \ln 3$$

$$c = \frac{4}{\ln 3}$$

3. [20 points] Suppose that it takes a force of 10 pounds to hold a certain spring 6 inches past its natural length. How much work, in foot-pounds, is required to stretch this spring from its natural length to 2 feet past its natural length?
(Hint: By Hookes Law, the force it takes to hold a spring stretched a distance x past its natural length is proportional to x.)

$F(x) = k(x)$

$F(x) = 20x$

$F(\frac{1}{2}) = 10 \text{ lb}$

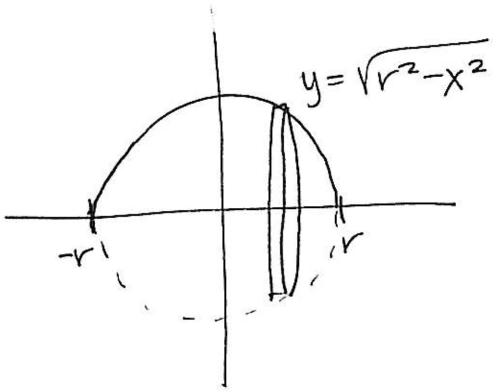
$k \cdot \frac{1}{2} = 10$

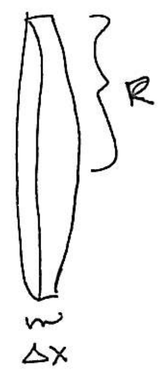
$k = 20$

$W = \int_0^2 20x \, dx = 10x^2 \Big|_0^2$

$= 10(4-0) = 40 \text{ ft-lb}$

4. [10 points] Find the volume of a solid sphere of radius r. Your answer should be in terms of r.
(Hint: This is the solid of revolution obtained by revolving the region between $y = 0$ and $y = \sqrt{r^2 - x^2}$ about the x-axis.)



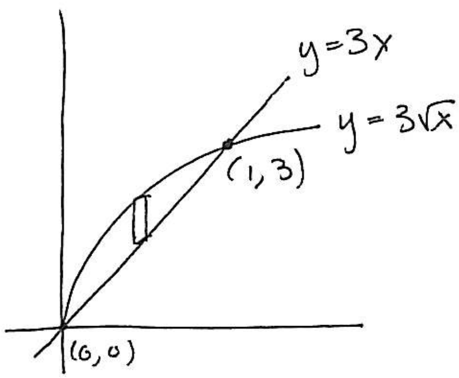


$$V_{disk} = \pi R^2 \Delta x$$
$$= \pi\left(\sqrt{r^2-x^2}\right)^2 \Delta x$$
$$= \pi\left(r^2-x^2\right)\Delta x$$

$$V = \int_{-r}^{r} \pi\left(r^2-x^2\right)dx$$
$$= \pi\left[x\cdot r^2 - \frac{x^3}{3}\right]_{-r}^{r}$$
$$= \pi\left[\left(r\cdot r^2 - \frac{r^3}{3}\right) - \left(-r\cdot r^2 - \frac{(-r)^3}{3}\right)\right]$$
$$= \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right)$$
$$= \frac{4\pi r^3}{3}$$

5. Find the volume of the solid of revolution obtained by revolving the region between the graphs of $y = 3x$ and $y = 3\sqrt{x}$ about the x-axis. Do so in two ways, to hopefully arrive at the same answer.

(a) [5 points] Use washers:



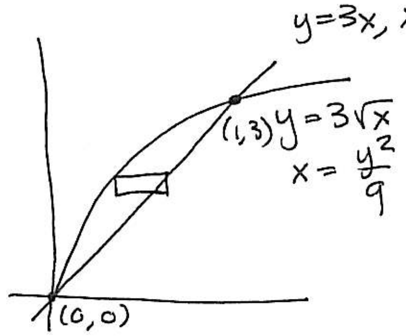
$$V_{washer} = \pi(R^2 - r^2)\Delta x$$
$$= \pi\left((3\sqrt{x})^2 - (3x)^2\right)\Delta x$$
$$= \pi(9x - 9x^2)\Delta x$$

$$3x = 3\sqrt{x}$$
$$x = \sqrt{x}$$
$$x = 0, 1$$

$$V = \int_0^1 \pi(9x - 9x^2)\,dx$$
$$= \pi\left[\frac{9}{2}x^2 - 3x^3\right]_0^1$$
$$= \pi\left[\left(\frac{9}{2}(1)^2 - 3(1)^3\right) - (0-0)\right]$$
$$= \frac{3}{2}\pi$$

(b) [5 points] Use shells:

$y=3x$, $x = \frac{y}{3}$



$$V_{shell} = 2\pi r h \Delta y$$
$$= 2\pi(y)\left(\frac{y}{3} - \frac{y^2}{9}\right)\Delta y$$
$$= 2\pi\left(\frac{y^2}{3} - \frac{y^3}{9}\right)\Delta y$$

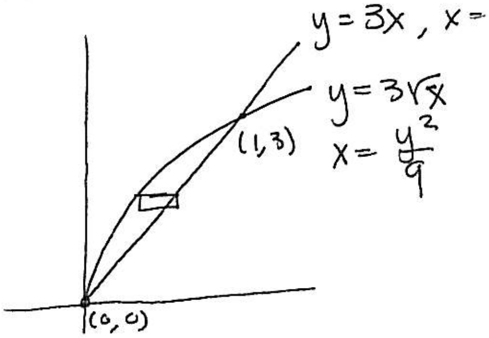
$$V = \int_0^3 2\pi\left(\frac{y^2}{3} - \frac{y^3}{9}\right)dy$$
$$= 2\pi\left(\frac{y^3}{9} - \frac{y^4}{36}\right)\Bigg|_0^3$$
$$= 2\pi\left(\frac{3^3}{9} - \frac{3^4}{36} - 0\right) = 2\pi\left(\cancel{3} - \frac{3^3}{4}\right) = 2\pi\left(\frac{\cancel{36} - 27}{12}\right)$$

$9 = 3^2 \qquad 36 = 4 \cdot 3^2$

$$= \frac{\cancel{2}\pi \cdot \cancel{9}^{3}}{\cancel{12}\,\cancel{4}\,2} = \frac{3\pi}{2}$$

6. Find the volume of the solid of revolution obtained by revolving the same region in #5, but now about the y-axis. Do so in two ways, hopefully arriving at the same answer, which is , however, smaller than your answer to #5.

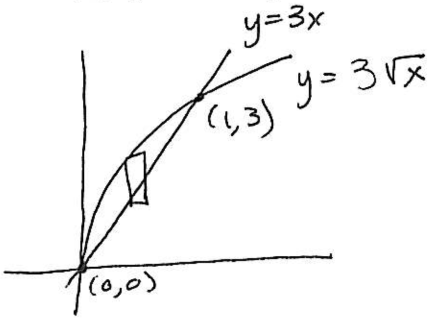
(a) [5 points] Use washers:



$$V_{washer}=\pi\left(R^2-r^2\right)\Delta y$$
$$=\pi\left(\left(\frac{y}{3}\right)^2-\left(\frac{y^2}{9}\right)^2\right)\Delta y$$
$$=\pi\left(\frac{y^2}{9}-\frac{y^4}{9^2}\right)\Delta y$$

$$V=\int_0^3\pi\left(\frac{y^2}{9}-\frac{y^4}{9^2}\right)dy$$
$$=\pi\left(\frac{y^3}{3^3}-\frac{y^5}{5\cdot 3^4}\right)\Big|_0^3=\pi\left(\frac{3^3}{3^3}-\frac{3^5}{5\cdot 3^4}-0\right)$$
$$=\pi\left(1-\frac{3}{5}\right)=\frac{2\pi}{5}$$

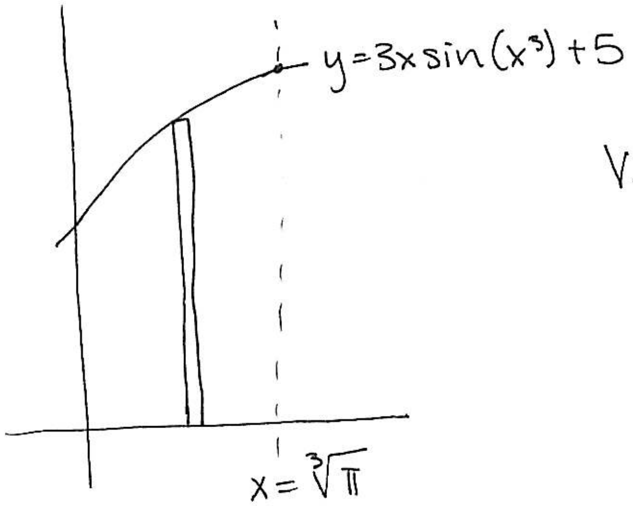
(b) [5 points] Use shells:



$$V_{shell}=2\pi r h\Delta x$$
$$=2\pi(x)\left(3\sqrt{x}-3x\right)\Delta x$$
$$=2\cancel{\pi}\left(3x^{3/2}-3x^2\right)\Delta x$$

$$V=\int_0^1 2\pi\left(3x^{3/2}-3x^2\right)dx$$
$$=2\pi\left[\frac{3x^{5/2}}{5/2}-x^3\right]_0^1=2\pi\left(\frac{3(1)^{5/2}}{5/2}-1^3-0\right)$$
$$=2\pi\left(\frac{2}{5}\cdot 3-1\right)=2\pi\left(\frac{6}{5}-1\right)=\frac{2\pi}{5}$$

7. [10 points] Use a method of your choice to find the volume of the solid of revolution obtained by revolving the region between the curves $y = 3x\sin(x^3) + 5$, $y = 0$, $x = 0$, and $x = \sqrt[3]{\pi}$, about the y-axis.



$$V_{shell} = 2\pi r h \Delta x$$
$$= 2\pi (x)(3x\sin(x^3)+5)\Delta x$$
$$= 2\pi \left(3x^2 \sin(x^3) + 5x\right)\Delta x$$

$$V = \int_0^{\sqrt[3]{\pi}} 2\pi\left(3x^2\sin(x^3) + 5x\right)dx$$

$$= 2\pi \int_0^{\sqrt[3]{\pi}} 3x^2 \sin(x^3)\,dx + 2\pi \int_0^{\sqrt[3]{\pi}} 5x\,dx$$

$u = x^3 \quad du = 3x^2dx$

$x=0 \to u=0 \quad x=\sqrt[3]{\pi} \to u=\pi$

$$= 2\pi \int_0^{\pi} \sin u\,du + 2\pi\left[\frac{5x^2}{2}\right]_0^{\sqrt[3]{\pi}}$$

$$= 2\pi\left[-\cos u\right]_0^{\pi} + 2\pi\left(\frac{5\pi^{2/3}}{2} - 0\right)$$

$$= 2\pi\left(-\cos\pi + \cos 0\right) + 5\pi^{5/3}$$

$$= 2\pi(1+1) + 5\pi^{5/3} = 4\pi + 5\pi^{5/3}$$

8. **Bonus Problems** (4 points each)

(a) What theorem guarantees that at least one such number c exists in 2(b)? Write down this theorem, including **all** the hypotheses, and also prove it, using the mean value theorem for derivatives and the Fundamental Theorem of Calculus.

* The Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof: If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is also continuous & diff on $[a,b]$, so the Mean Value Thm for derivatives applies:

There is a c in $[a,b]$ such that

$$F'(c) = \frac{F(b)-F(a)}{b-a}.$$

But $F'(c) = \left(\int_a^x f(t)\,dt\right)'(c) = f(c)$ and

$$\frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\left[\int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt}^{\,0}\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)\,dt = f_{ave}$$

So $f(c) = f_{ave}$. ✓

(b) Let $f(t) = t\sin(t^2)$, and let $g(x)$ be the average value of $f(t)$ from $t = 0$ to $t = x$. What is $g'(x)$? Your answer should be in terms of x only.

$$g(x) = \frac{1}{x-0}\int_0^x t\sin(t^2)\,dt$$

$$= \frac{1}{x}\int_0^x t\sin(t^2)\,dt \qquad u=t^2 \quad du = 2t\,dt$$

$$t=0 \to u=0 \qquad t=x \to u=x^2$$

$$= \frac{1}{x}\int_0^{x^2} \frac{1}{2}\sin(u)\,du$$

$$= \frac{1}{2x}\Big[-\cos u\Big]_0^{x^2} = \frac{1}{2x}\left(-\cos(x^2)+\cos(0)\right)$$

$$= \frac{1-\cos(x^2)}{2x}$$

$$g'(x) = \frac{(2x)(-2x\sin(x^2)) - (1-\cos(x^2))(2)}{(2x)^2}$$

$$= \frac{-4x^2\sin(x^2) - 2 + 2\cos(x^2)}{4x^2}$$

$$= \frac{\cos(x^2)-1}{2x^2} - \sin(x^2)$$