Integration Review Sheet

1. Formulas For Some Familiar Integrals

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln a}a^x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \tan x\,dx = \ln|\sec x| + C$

$\int \cot x\,dx = \ln|\sin x| + C$

$\int \frac{1}{1+x^2}\,dx = \arctan x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

2. Integration By Substitution (Indefinite Integrals)

Suppose that we have an integral of the form $\int f(g(x))g'(x)dx$. That is, we can (roughly) identify an "inside" function (here, it's g), and its derivative (g'). Then, if we set $w = g(x)$ (and consequently $dw = g'(x)dx$), we have that

$$\int f(g(x))g'(x)dx = \int f(w)dw.$$

If we're able to identify dw up to a constant factor, then the integral may still be computed by simultaneously multiplying and dividing by the desired constant. For example, if we want to compute

$$\int \theta\cos\left(\theta^2\right)d\theta,$$

we may choose $w = \theta^2$, $dw = 2\theta d\theta$. However, we're off by a factor of 2. We may compute the integral via the following manipulations:

$$\begin{aligned}\int \theta\cos\left(\theta^2\right)d\theta &= \int \frac{1}{2}\cdot 2\theta\cos\left(\theta^2\right)d\theta \\ &= \frac{1}{2}\int \cos\left(\theta^2\right)2\theta d\theta.\end{aligned}$$

We now make our substitution to get

$$\begin{aligned}\int \theta\cos\left(\theta^2\right)d\theta &= \frac{1}{2}\int \cos\left(\theta^2\right)2\theta d\theta \\ &= \frac{1}{2}\int \cos\left(w\right)dw = \frac{1}{2}\sin w + C \\ &= \frac{1}{2}\sin\left(\theta^2\right) + C.\end{aligned}$$

3. Integration By Substitution (Definite Integrals)

Suppose that we have a definite integral of roughly the same form as above, namely $\int_a^b f(g(x))g'(x)dx$. That is, we can identify an "inside" function (here, it's g), and its derivative (g'). Then, if we set $w = g(x)$ (and consequently $dw = g'(x)dx$), we have that

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(w)dw.$$

Like before, it's still okay if we're off by a constant factor. Also, note that once we convert to our new variable and find our new bounds, we don't need to return to our original variable or our original bounds. We may do the new integral as if the original one never existed.

As an example, suppose we want to find

$$\int_1^e \frac{\ln t}{t}\, dt.$$

Taking the substitution $w = \ln t$, (and so $dw = \frac{1}{t}dt$), we compute our new bounds:

- upper: $\ln(e) = 1$
- lower: $\ln(1) = 0$

Thus our new integral is

$$\int_0^1 w\, dw.$$

This is simple enough to compute: our antiderivative is $\frac{1}{2}w^2$, so this new integral evaluates to

$$\int_0^1 w\, dw = \frac{1}{2}w^2\Big|_0^1$$

$$= \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}.$$

4. Integration By Parts (Indefinite Integrals)

Suppose we have an integral that involves the product of two functions. If we can identify functions f and g and write the integral as $\int f(x)g'(x)dx$, then we have that

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

If we use the notation $u = f(x)$ and $v = g(x)$, then we have $du = f'(x)dx$ and $dv = g'(x)dx$, and the above equation may be written as

$$\int u\, dv = uv - \int v\, du.$$

In this fashion, we need only identify u and dv in order to compute the integral. It's worth noting that the first choice of u and dv may not be the correct one. Make sure that when you choose dv, you should be able to (fairly) easily compute v, and that the new integral you get is less complicated than the first one.

As an example, suppose we want to compute

$$\int x \sin x \, dx.$$

There are several choices for u and dv, and it's worth writing some down just to get a feel for it. For now, I'll choose $u = x$ and $dv = \sin x \, dx$. We easily compute $du = dx$ and $v = -\cos x$. Thus, we have

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x - \int -\cos x \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C. \end{aligned}$$

5. Integration By Parts (Definite Integrals)

Integration by parts changes little when we want to compute definite integrals. Like before, we have an integral that involves the product of two functions. If we can identify functions f and g and write the integral as $\int_a^b f(x)g'(x)dx$, then

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\, dx.$$

This is the same formula as in the indefinite case, with two alterations. First of all, both integrals shown are definite and have the same bounds. Secondly, the product $f(x)g(x)$ is followed by the vertical bar $|_a^b$, which means that we evaluate first at b, and then subtract off what we get by evaluating at a. This is exactly the same notation that we've been using for definite integrals in the past. If we wanted, we could write this as $f(b)g(b) - f(a)g(a)$ instead of $f(x)g(x)|_a^b$; either one is correct.

As an example, suppose that we want to evaluate the definite integral

$$\int_0^1 we^w \, dw.$$

We choose $u = w$ and $dv = e^w dw$, which gives us $du = dw$ and $v = e^w$. The formula then tells us that

$$\int_0^1 we^w \, dw = we^w\Big|_0^1 - \int_0^1 e^w \, dw.$$

This is straightforward to evaluate:

$$\begin{aligned} we^w\Big|_0^1 - \int_0^1 e^w \, dw &= (1 \cdot e^1 - 0 \cdot e^0) - \left(e^w\Big|_0^1\right) \\ &= e - (e^1 - e^0) = e - e + 1 = 1. \end{aligned}$$

So we conclude that $\int_0^1 we^w dw = 1$.