INVERSE TRIGONOMETRIC FUNCTIONS

1. Inverse Functions

Intuitively, a function $f(x)$ has an inverse, if the function can be undone. That is, if for any y we want to find the value of x for which $f(x) = y$ and we get one and only one value of x, the function is said to have an inverse.

Example 1. *Let's start by looking at the functions $f(x) = x^3$ and $g(x) = x^2$. Suppose I wanted to find the value x that gave $f(x) = 64$ and $g(x) = 64$. Notice that in the case of $f(x)$ there is one and only one x value that gives $f(x) = 64$, namely $x = 4$. On the other hand, for $g(x)$ there are two possible values of x that $g(x) = 64$, namely $x = \pm 8$. So, we see that the function $f(x)$* ***does*** *have an inverse (there is, after all, nothing special about the value $y = 64$) while the function $g(x)$* ***does not*** *have an inverse.*

There is a way you can tell if a function has an inverse by looking at its graph. First recall the **Vertical Line Test** which says that a graph is a function if any line drawn parallel to the y-axis intersects the graph at most once. From this you can see that in Example 1 both $f(x)$ and $g(x)$ are functions whereas the graph of $y = \pm\sqrt{x}$ is not a function.

Suppose now that we can graph a function $f(x)$ and want to graph its inverse. Remember to find the inverse of a function we find the x-values that come from a given y-value. Notice this is switching the normal roles of x and y: normally we find y values from a given x value. Graphically, we can model this by switching the x and y coordinates of the points on the graph. For this graph to be an inverse, this graph has to be a function, i.e., it has to pass the Vertical Line Test.

Example 2. *Like in Example 1 let $f(x) = x^3$ and $g(x) = x^2$. We already know that $f(x)$ does have an inverse function (since we get one and only one value for each fixed y) and the $g(x)$ does not since at least one y value could come from two places. Let's check these results with our graphical test. Notice, first, that switching x and y coordinates is the same as reflecting the graph across the line $y = x$ (you should convince yourself of this). The reflected graph of $f(x) = x^3$ looks like a stretched-out S (see Figure 2) and passes the Vertical Line Test and so is an inverse. On the other hand, the reflected graph of $g(x)$ is a parabola rotated 90 degrees to the right(see Figure 1). This is not an inverse since it fails the vertical line test.*

If a function $f(x)$ has an inverse we denote the inverse by $f^{-1}(x)$.

Notation. *Realize that $f^{-1}(x)$ is not the same as $[f(x)]^{-1} = \frac{1}{f(x)}$.*

Now that you have this intuition we can make this

Definition 1. Let $f : A \to B$ be a function where A and B are subsets of $\mathbb{R}$. If there exists another function $f^{-1}(x) : B \to A$ for which

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

then $f^{-1}(x)$ is called the **inverse function** of $f(x)$.

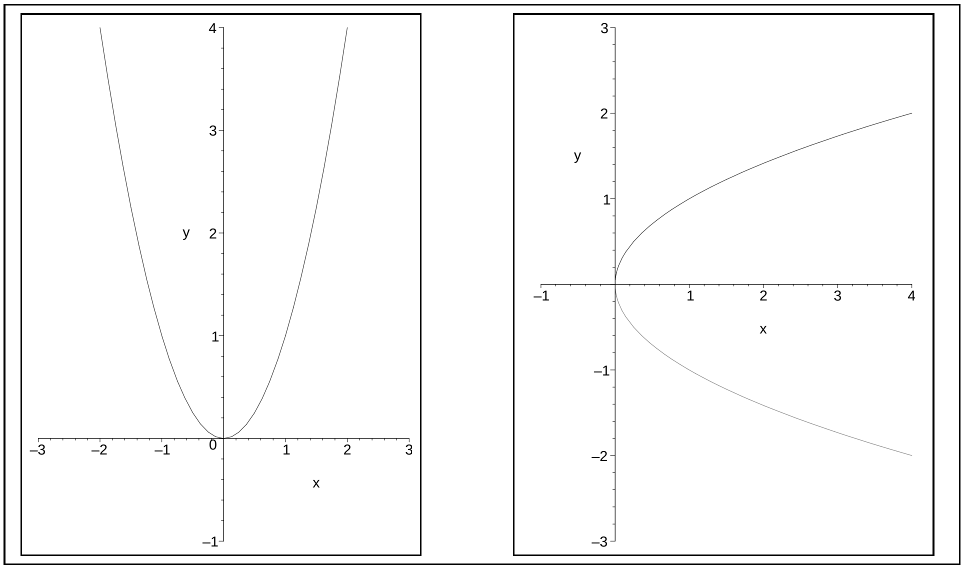

FIGURE 1. On the left is the graph of $g(x) = x^2$ and on the right is that graph reflected around the line $y = x$

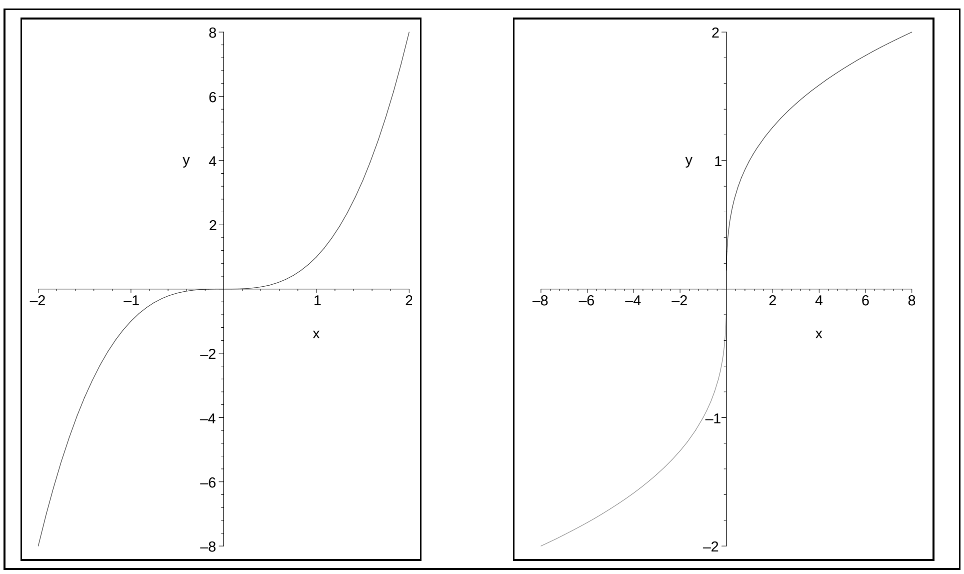

FIGURE 2. On the left is the graph of $f(x) = x^3$ and on the right is that graph reflected around the line $y = x$

Since the Vertical Line Test tells us whether or not a graph is a function, it would be nice to have a similar test that tells us more quickly whether or not the graph of a function has an inverse. Let's examine what we have been doing thus far. We have a graph, reflect it across the line $y = x$ and apply the Vertical Line Test. If, on the other hand, we were to draw a horizontal line on the original, nonreflected graph, we would get the same information. This is because when we reflect the graph and the horizontal line across $y = x$ we get a vertical line going through the reflected graph. The number of times this horizontal line intersects the original graph, is the same as the number of times the resulting vertical line intersects the reflected graph. In short, the **Horizontal Line Test** tells us a quick way to tell if a graph has an inverse: if any horizontal line intersects a graph no more than once, the function that generated that graph has an inverse.

Example 3. *Before we go on, we note that while $g(x) = x^2$ does not have an inverse function (see Example 2), if we define $G(x) = x^2$ for positive x we see that $G(x)$ **does** have an inverse, since when $G(x)$ passes the Horizontal Line Test. In general, then, if we can find a part of the graph that does pass the Horizontal Line Test, and restrict our attention to that domain, we can invert functions.*

Now, as the title suggests, we will investigate $\sin(x)$, $\cos(x)$, $\tan(x)$ and their inverses. We'll treat the case of $\sin(x)$ intuitively and expect you to do the same with $\cos(x)$ and $\tan(x)$.

Since $\sin(x)$ does not pass the Horizontal Line Test (see Figure 3), it does not have an inverse. All is not lost, however, because of what we did in Example 3. Remember, in the example we limited our attention to a part of $f(x)$'s domain that would have an inverse: i.e., a part that would pass the Horizontal Line Test. There are several parts to which we could limit our attention. For example, x values between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, or x values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, etc. By convention, we limit our attention to the first interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. So, we define $\text{Sin}(x) = \sin(x)$ for x in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. This function $\text{Sin}(x)$ has an inverse and we denote it by $\text{Sin}^{-1}(x)$.

Similar arguments and conventions allow us to make this

Definition 2. The inverse trigonometric functions $\text{Sin}^{-1}(x), \text{Cos}^{-1}(x)$ and $\text{Tan}^{-1}(x)$ are defined as

- $y = \text{Sin}^{-1}(x)$ for x in $[-1, 1]$ if and only if $\sin(y) = x$ for y in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.
- $y = \text{Cos}^{-1}(x)$ for x in $[-1, 1]$ if and only if $\cos(y) = x$ for y in $[0, \pi]$.
- $y = \text{Tan}^{-1}(x)$ for x in $(-\infty, \infty)$ if and only if $\text{Tan}(y) = x$ for y in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let's do some examples:

Example 4. *Evaluate each expression:*

(1) $Sin^{-1}(0)$,

(2) $Sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$.

(3) $Sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)$.

(4) $Sin^{-1}\left(\sin\left(\frac{-5\pi}{6}\right)\right)$.

Solution. (1) *Let $y = Sin^{-1}(0)$ for y in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Then $\sin(y) = 0$ so $y = 0$.*

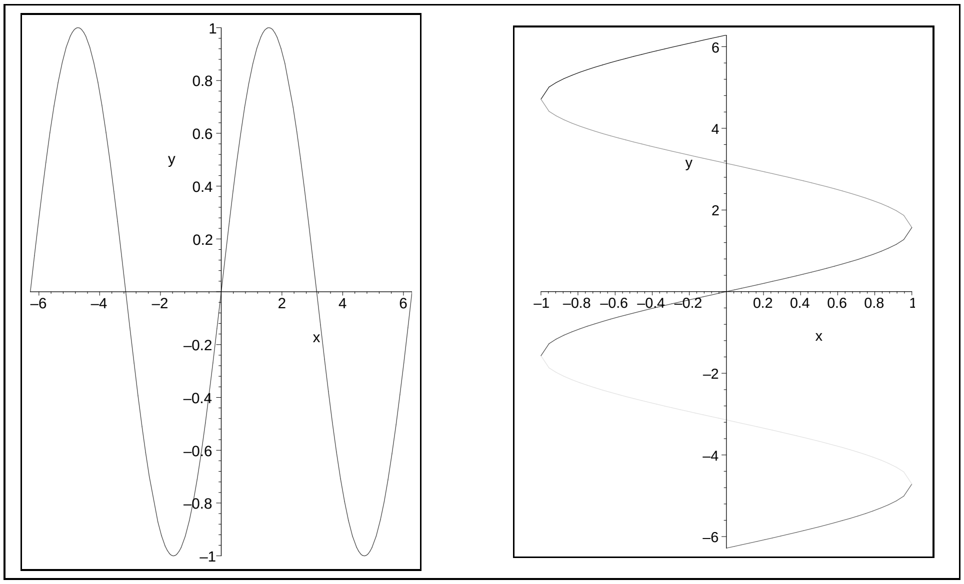

FIGURE 3. On the left is the graph of $h(x) = \sin(x)$ and on the right is that graph reflected around the line $y = x$

(2) *Similarly,* $Sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

(3) *Note that* $\sin\left(\frac{-\pi}{6}\right) = \frac{-1}{2}$. *Then* $Sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right) = Sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$.

(4) *Almost identically, we get* $Sin^{-1}\left(\sin\left(\frac{-5\pi}{6}\right)\right) = \frac{-\pi}{6}$.

Remark. Notice for the last two examples above we entered different values into the function, but got the same thing out. This is an artifact of the way we limited our domain. Notice, also, that what these inverse functions do is take a ratio and give an angle.

Example 5. *Show that* $\cos\left(Sin^{-1}(x)\right) = \sqrt{1-x^2}$.

Solution. *Let* $y = Sin^{-1}(x)$ *for* y *in* $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. *Then* $\cos\left(Sin^{-1}(x)\right) = \cos(y)$. *Since* y *is either in the first or fourth quadrant,* $\cos(y) \geq 0$. *Using the Pythagorean Identity*

$$\cos^2(y) + \sin^2(y) = 1$$

and the fact that for this particular $\sin(y) = x$ *we obtain*

$$\cos\left(Sin^{-1}(x)\right) = \cos(y) = +\sqrt{1-\sin^2(y)} = +\sqrt{1-x^2}.$$

We chose the positive part of the square root function because of the way we chose the y *at the beginning of the example.*

Remark. Remember that $\cos^2(x) = [\cos(x)]^2$ but $\text{Cos}^{-1}(x) \neq [\text{Cos}(x)]^{-1}$.

Remark. Other notation for $\mathrm{Sin}^{-1}(x)$ is commonly used. E.g., $\mathrm{Sin}^{-1}(x)$ is often written as one of $\sin^{-1}(x)$, $\mathrm{Arcsin}(x)$ or $\arcsin(x)$.

2. Problems

Evaluate without a calculator

(1) $\mathrm{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$
(2) $\mathrm{Tan}^{-1}(1)$
(3) $\mathrm{Cos}^{-1}(1)$
(4) $\mathrm{Sin}^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
(5) $\cos\left(\mathrm{Sin}^{-1}\left(\frac{3}{5}\right)\right)$
(6) $\tan\left(\mathrm{Cos}^{-1}\left(\frac{2}{3}\right)\right)$

Express without using trig or inverse trig functions (see Example 5)

(1) $\sin(\mathrm{Cos}^{-1}(x))$
(2) $\sec\left(\mathrm{Cos}^{-1}\left(\frac{1}{x}\right)\right)$.