DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. Implicit differentiation

In math 1, we learned about the function $\ln x$ being the inverse of the function e^x. Remember that we found the derivative of $\ln x$ by differentiating the equation

$$\ln x = y.$$

First, you wrote it in terms of functions that we knew:

$$x = e^y$$

Then, we took the derivative of both sides

$$1 = e^y \frac{dy}{dx}.$$

Then, since $e^y = x$, we simplified to

$$1 = x\frac{dy}{dx}$$

and concluded by dividing both sides by x to get

$$\frac{1}{x} = \frac{dy}{dx}.$$

2. Inverse trig functions

We will do the same for the inverse trig functions. The process is the same, it is just a little hard to simplify.

Example 1. *Find the* $\frac{dy}{dx}$ *when* $y = Sin^{-1}(x)$.

Solution. *Again we start by writing it in terms of functions we know better, so*

$$\sin(y) = x$$

for $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. *Now, take the derivative of both sides,*

$$\cos(y)\frac{dy}{dx} = 1.$$

Now $y = Sin^{-1}(x)$ *so we need to simplify* $\cos(Sin^{-1}(x))$. *We did this in Example 5 of the previous packet where we showed*

$$\cos(Sin^{-1}(x)) = \sqrt{1 - x^2}.$$

So we conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Patterning our work after the example we can show that

(1) for $y = \text{Tan}^{-1}(x)$, we get $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

(2) for $y = \text{Cos}^{-1}(x)$, we get $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

3. Problems

Repeat the Example for

(1) $y = \text{Tan}^{-1}(x)$

(2) $y = \text{Cos}^{-1}(x)$

Find the derivatives of the following functions

(1) $f(x) = \text{Sin}^{-1}(2x-1)$.

(2) $h(x) = (1+x^2)\text{Tan}^{-1}(x)$.

(3) $y = \frac{cos^{-1} t}{t}$.

(4) $g(x) = \text{Tan}^{-1}(\sin(x))$.

(5) $y = \text{Tan}^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$.

(6) $F(t) = \sqrt{1-t^2} + \text{Sin}^{-1} t$.

(7) $f(x) = x \sin x \text{Cos}^{-1} x$

(8) $y = (\text{Sin}^{-1} x)^2$

(9) $y = \text{Sin}^{-1} x^2$

(10) $U(t) = e^{\text{Tan}^{-1} t}$.

Solutions to the odd numbered ones of the last 10:

- (1) $\frac{2}{\sqrt{1-(2x-1)^2}}$
- (3) $\frac{\frac{-t}{\sqrt{1-t^2}} - \cos^{-1} t}{t^2}$
- (5) $\frac{\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2} + \frac{\frac{1}{2}\left(\frac{x-a}{x+a}\right)^{\frac{-1}{2}}\left(\frac{(x+a)-(x-a)}{(x+a)^2}\right)}{\sqrt{\frac{x-a}{x+a}}}$
- (7) $\sin x \cos^{-1} x + x\left(\cos x \cos^{-1} x - \frac{\sin x}{\sqrt{1-x^2}}\right)$
- (9) $\frac{2x}{\sqrt{1-x^4}}$