Chapter 7 Supplemental Notes
Math 29 Spring 2011

1. Index set examples

1. Fin is Σ_2-complete. Step 1: show it is in Σ_2.

$$\text{Fin} = \{e : |W_e| < \infty\} = \{e : (\exists N)(\forall n)(\forall s)\,(n > N \Rightarrow \varphi_{e,s}(n)\uparrow)\}$$

This gives us the chance to see why duplicates of the same quantifier don't affect complexity. If we wanted, we could use the following formula:

$$(\exists N)(\forall x)\,[(\forall n < x)(\forall s < x)\,(n > N \Rightarrow \varphi_{e,s}(n)\uparrow)]$$

The part in square brackets is still a computable relation, and this formula is true if and only if the original version is true.

We will come back to Step 2: Every Σ_2 set 1-reduces to Fin.

2. Tot is Π_2-complete. Step 1:

$$\text{Tot} = \{e : W_e = \mathbb{N}\} = \{e : (\forall n)(\exists s)(\varphi_{e,s}(n)\downarrow\}$$

Step 2 will happen in conjunction with Step 2 for Fin.

3. Rec is Σ_3-complete. Rec $= \{e : W_e$ is computable$\}$, so the leading existential quantifier says that there must be an index $\hat{e}$ giving W_e's characteristic function. For $\varphi_{\hat{e}}$ to be that function, it must be total and have output 1 on n if $n \in W_e$ and output 0 otherwise. Going from the "not in W_e" side is hard, though, because we only have an enumeration, so we need to make sure $\varphi_{\hat{e}}$ is total with codomain $\{0,1\}$ and membership in W_e is equivalent to an output of 1.

For every n, if n appears in W_e at some stage we must have $\varphi_{\hat{e}}(n)\downarrow= 1$, but not necessarily at the same stage. Likewise halting with output 1 says n must appear in W_e eventually, but not at a specific stage.

$$\text{Rec} = \{e : (\exists \hat{e})(\forall n, s)(\exists t)[(\varphi_{\hat{e},t}(n)\downarrow\in \{0,1\}\ \&$$
$$(n \in W_{e,s} \Rightarrow \varphi_{\hat{e},t}(n) = 1)\ \&\ (\varphi_{\hat{e},s}(n) = 1 \Rightarrow n \in W_{e,t})]\}$$

We omit Step 2 of the completeness proof.

That Inf is Π_2-complete comes from the proof for Fin. Showing Con is Π_2 is homework, and that it is actually complete will follow, with Tot, from the completeness proof for Fin. Note that because the *S-m-n* Theorem produces a 1-1 computable function, the proof that K is Turing-complete shows it is Σ_1-complete as well.

Step 2. Showing completeness.

For any given Σ_2 set A, we must produce a computable 1-1 function f such that $x \in A \Leftrightarrow f(x) \in \text{Fin}$. For some computable relation R, $x \in A \Leftrightarrow (\exists y)(\forall z)R(x,y,z)$, by definition of being Σ_2. It turns out to be more useful to take the complement: $x \in \overline{A} \Leftrightarrow (\forall y)(\exists z)\neg R(x,y,z)$, because we can then "cap off" the leading universal quantifier at higher and higher points, looking to see if there is a z for each of the finitely many y. This allows us to define a partial computable function:

$$\psi(x,w) = \begin{cases} 0 & (\forall y \le w)(\exists z)\neg R(x,y,z) \\ \uparrow & \text{otherwise} \end{cases}$$

This is partial computable despite the unbounded existential quantifier because we can dovetail the (finitely-many) searches for a z to match each $y \le w$. If all y have such a z we will eventually find it, and if not, it is simply a divergent unbounded search. As usual, ψ is some φ_e, we can use *s-m-n* to push x into the index, e is fixed by A and we end up with a 1-1 total computable f such that $\varphi_{f(x)}(w) = \psi(x,w)$.

If $x \in \overline{A}$, all y have a matching z, every w gives a convergent search, and $\varphi_{f(x)}$ is the constant 0 function. If $x \in A$, there is some y that has no matching z, and $\varphi_{f(x)}$ will diverge on all w that surpass the least such y. That is,

$$x \in A \;\Rightarrow\; W_{f(x)} \text{ finite } \;\Rightarrow\; f(x) \in \text{Fin}, \text{ and}$$
$$x \in \overline{A} \;\Rightarrow\; (\forall w)(\varphi_{f(x)}(w) = 0) \;\Rightarrow\; f(x) \in \text{Con} \subset \text{Tot} \subset \text{Inf} = \overline{\text{Fin}}.$$

2. A picture of the arithmetic hierarchy

Each set is contained in those directly above it and those above it to which it is connected by lines.