HOMEWORK ASSIGNMENT #3, DUE MONDAY, 10/18/2010

Notice the slightly later due date! This is because the first midterm is on Thursday; the 4th homework assignment will be due on Friday, as usual, though. Remember to write clearly and to justify all your claims in your solutions.

(1) Show that there are infinitely many primes of the form $6k+5$. (Obviously, you are not supposed to cite Dirichlet's Theorem or anything of that sort. You should be using the methods in Euclid's proof for the infinitude of prime numbers and the modification of the method for primes of the form $4k+3$.)

(2) Prove the following generalization of the fact that $\sqrt{2}$ is irrational: if k is a positive integer, and n is not a perfect kth-power (ie, if $\sqrt[k]{n}$ is not an integer), then $\sqrt[k]{n}$ is irrational.

(3) Let n be a positive integer and let N be the least common multiple of $1, 2, \ldots, n$. Show that the sum of the exponents appearing in the prime factorization of N is always less than n.

(4) Recall that for integers n, m satisfying $0 \leq m \leq n$, the binomial coefficient $\binom{n}{m}$ is defined to be the expression

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

For instance, $\binom{4}{2} = 4!/(2!2!) = 24/4 = 6$, while $\binom{7}{0} = 7!/(7!0!) = 1$ and $\binom{5}{3} = 5!/(3!2!) = 120/12 = 10$. This number is equal to the number of ways of choosing m objects from a set of n objects, and so is an integer. If p is a prime and $1 < k < p$, show that $\binom{p}{k}$ is divisible by p.

(5) Recall that we can test a positive integer for divisibility by 2 if its last digit is even, and for divisibility by 4 by checking if its last two digits form a number divisible by 4. Show that we can test a positive integer for divisibility by 2^k by checking if the number formed by its last k digits is divisible by 2^k.

(6) Find all solutions (there may be none) of the following congruences. Be sure to explain why your answer is correct.

(a) $5x \equiv 7 \mod 12$.

(b) $21x \equiv 13 \mod 105$.

(c) $9x \equiv 6 \mod 15$.

(d) $x^2 \equiv 1 \mod 11$.

(e) $x^2 \equiv 1 \mod 8$.

(7) Find last digit of the following numbers. The work you show should not assume the use of a calculator or other computational device at any point.

(a) 7^{7143},

(b) $23! + 19! + 15! + 11! + 7! + 3!$,

(c) $2 \uparrow\uparrow n$, for $n \geq 3$, where $a \uparrow\uparrow n$ means a power tower of a with size n: for instance, $2 \uparrow\uparrow 3 = 2^{2^2} = 2^4$, while $2 \uparrow\uparrow 4 = 2^{2^{2^2}} = 2^{2^4} = 2^{16}$. (Remember that towers of exponentials are evaluated from the top down, not the bottom up, so for instance $3^{3^3} = 3^{27}$, not $(3^3)^3 = 27^3$, which is a much smaller number than 3^{27}.