HOMEWORK ASSIGNMENT #2, DUE FRIDAY, 10/8/2010

Remember to write clearly and to justify all your claims in your solutions.

(1) Let $a_1, \dots, a_n$ be nonzero integers. For what values of d does the equation $a_1x_1 + \dots + a_nx_n = d$ have integer solutions $x_1, \dots, x_n$?

(2) Let $a_1, \dots, a_n$ be positive integers. Show that $\operatorname{lcm}(a_1, \dots, a_n) = \operatorname{lcm}(\operatorname{lcm}(a_1, a_2), a_3, \dots, a_n)$. (This is the mirror of exercise 1.9 in the textbook.)

(3) Find all integer solutions (x, y) to the equation $192x + 66y = 12$.

(4) The post office in Integerland issues 20 cent and 12 cent stamps. You want to make exactly $2.72 in postage from these stamps. List all the different ways you can do so, and prove that your answer is correct. (Remember, you can't use a negative number of stamps.)

(5) (This problem is worth double.) Now suppose Integerland issues stamps in a and b cent denominations, where a, b are relatively prime positive integers both greater than 1. Show that
  (a) $ab - a - b > 0$.
  (b) Show that the equation $ax + by = ab - a - b$ has integer solutions x, y, but no matter how hard you try, you cannot actually make $ab - a - b$ cents worth of postage using stamps of size a, b (this means that you should show that any integer solution x, y of $ax + by = ab - a - b$ satisfies either $x < 0$ or $y < 0$), and
  (c) if d is any integer with $d > ab - a - b$, then you can make d cents worth of postage from some combination of a and b cent stamps.

(6) We know that if p is a prime, n any positive integer, and $p|a^n$, then $p|a$. Classify all numbers d such that for any positive n, if $d|a^n$, then $d|a$. (You should give a simple description of all d which satisfy this property, prove that all such d do indeed satisfy this property, and then give a counterexample for each d which does not satisfy your description.)

(7) Recall that if n is a positive integer, $n! = n(n-1)(n-2)\dots(2)(1)$ is the product of the first n positive integers.
  (a) Let m be a positive integer. Show that the number of integers between 1 and n which are divisible by m is equal to $\lfloor n/m \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. For instance, $\lfloor 3 \rfloor = 3$, $\lfloor e \rfloor = 2$, $\lfloor 13/2 \rfloor = 6$.
  (b) Let p be a prime. Show that $v_p(n!)$ (that is, the exponent of the highest power of p dividing $n!$) is given by the formula

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \left\lfloor \frac{n}{p^1} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$$