A Guide to the Use of Math Symbols and Techniques

for use in Math 25, Fall 2006

I. Guide to Math Symbols

So far, we've been using some symbolic notation to express certain mathematical ideas, and many of them may be new to you. What I hope to do here is give a concise list of these terms, and explain how they will be used in this course.

Before we get started, I want to briefly review the mathematical concept of a set. Basically, a *set* is any collection of objects (finite or infinite). An object that is among those in the set is called an *element*. For most of this course, our sets will be those with integer elements.

Examples:

- $\mathbb{Z}$: the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- $\mathbb{N}$: the set of natural numbers: $\{1, 2, 3, \ldots\}$. Often we'll use the notation $\mathbb{Z}^+$, as $\mathbb{N}$ is also the set of positive integers.
- $\in$: means "contained in" or "is contained in." We use this to describe when something is an element of a set. Analagously, we use $\notin$ to indicate that something is not the element of a particular set. For example, $-3 \in \mathbb{Z}$ but $-3 \notin \mathbb{Z}^+$.
- set notation : this one gets used a lot, by both myself and your text. Basically, it's a way of describing what kind of elements are contained in a set. For example, if n and m are integers, consider
$$\{an + m : a \in \mathbb{Z}^+\}.$$
We break down this into pieces as:
  - "$\{$" = "the set of all things"
  - "$:$" = "such that" or "for which" (sometimes a vertical bar "$|$" will be used instead).

  Thus, the expression above reads as "the set of all things of the form $an + m$ for which a is a positive integer."
- $\subset$: also means "contained in" or "is contained in," but with respect to a set contained in another set. For example, $\mathbb{Z}^+ \subset \mathbb{Z}$. (You may also see "$\subseteq$," which carries the same meaning for our purposes.)
- s.t. : means "such that." This is a common shorthand, as in "let n be an integer s.t. $n > 5$."
- $\forall$: means "for all" or "for every." As in: "$\forall$ integer(s) n, either $n \geq 0$ or $n < 0$."
- $\exists$: means "there exists." As in: "$\exists$ a positive integer n for which $n < \sqrt{5}$." If the integer in question is unique, you may see $\exists!$, as in "$\exists!$ a positive integer n for which $n < \sqrt{2}$."

- i.e. : this is from a Latin phrase, "id est," which means "that is." As in "…so $n \in \mathbb{N}$; i.e., n is a positive integer."
- e.g. : comes from another Latin phrase "exempli gratia," which means "for example."
- $\square$ or $_{//}$: I typically use these symbols to indicate that I've finished proving something.
- $\therefore$: this one may not get as much use; it means "therefore."
- $\implies$: this means "implies," as in "n even $\implies$ 2 divides n." Think of this as equivalent to saying "If n is even, then 2 divides n."
- iff : this means "if and only if," as in "n is a positive integer if and only if it is a natural number."

As always, other symbols may come up over the course of the term, either introduced by the textbook or myself. If you're confused about the meaning of a symbol (either in this handout or somewhere else), please don't hesitate to ask!

II. Proof Techniques

1. Proof by Induction

Induction will be the most common proof technique that we will use in this course. Your textbook gives two variants of proof by induction, which I will summarize here:

- First principle of mathematical induction ("Weak induction"). We would like to prove that a certain statement is true for a particular set, usually the set of positive integers, but occasionally the set of nonnegative integers. Suppose that we wish to prove that

  $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

  for all positive integers n. We can prove it inductively as follows:

  1. **Base case.** Take the case $n = 1$. (If we were proving a statement for nonnegative integers, we would begin with zero.) Simply check that the statement holds for $n = 1$:

     $$\sum_{k=1}^{1} k^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6} = \frac{1(1+1)(2+1)}{6}.$$

  2. **Inductive case.** We begin this step by stating the *inductive hypothesis*. Specifically, this is "Suppose that $n \geq 1$ is arbitrary, and that the statement holds for n." We want

to show that given our inductive hypothesis, it follows that the statement is true for $n+1$ as well. This may vary from proof to proof, but here we may proceed as follows:

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2 \\ &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6},\end{aligned}$$

where the last equality holds because of the inductive hypothesis. We now play around with some algebra:

$$\begin{aligned}(n+1)^2 + \frac{n(n+1)(2n+1)}{6} &= (n+1)\left((n+1) + \frac{n(2n+1)}{6}\right) \\ &= (n+1)\left(\frac{6(n+1)+2n^2+n}{6}\right) \\ &= (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\ &= (n+1)\left(\frac{(n+2)(2n+3)}{6}\right) \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},\end{aligned}$$

which proves the statement for $n+1$. Thus, the statement in question is true for all positive integers.

- Second principle of mathematical induction ("Strong induction"). As before, we would like to prove that a certain statement is true for a particular set, usually the set of positive integers, but occasionally the set of nonnegative integers. Let's again prove that the statement

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for all positive integers n.

1. **Base case.** As before, we begin by proving the statement for the case $n = 1$. This is identical to the previous proof, so we omit it here.
2. **Inductive case.** The inductive hypothesis is slightly different: "Suppose that $n \geq 1$ is arbitrary and that the statement holds for all positive integers $1, 2, \ldots, n$." We proceed by proving the statement for $n+1$, in much the same way as before:

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2 \\ &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

The rest of the proof follows as before.

As is clear, the difference between "weak" and "strong" induction is not very great, and many mathematicians (including the author of your textbook) choose not to use the terms "weak" and "strong" at all, as this tends to imply that strong induction is more powerful, which it in general is not. As it turns out, strong induction is used more commonly in computer science, though we will make use of both in this course. As a final note, we have seen in class that a modified version of strong induction is more useful for second-order recurrences (e.g., the Fibonacci sequence), provided we make a change to the base case.

2. Proof by Contradiction

This is another useful technique that we will use a great deal in this course. Effectively, if we want to prove that "Statement A implies Statement B," we may prove by contradiction by (1) assuming Statement A is true and Statement B is false simultaneously, then (2) showing that this assumption gives rise to a logical contradiction. Thus, we conclude, it must be the case that Statement A implies Statement B. An example: suppose we want to prove "$\sqrt{2}$ is irrational"; that is, "$\sqrt{2}$ is not equal to a fraction of integers $\frac{a}{b}$." Note that this is not quite a "Statement A implies Statement B" situation. However, if we say "If $\sqrt{2}$ is the positive root of the polynomial $x^2 - 2$, then $\sqrt{2}$ is not equal to a fraction of integers $\frac{a}{b}$," then it fits our mold. The proof:

1. Suppose that $\sqrt{2}$ is defined as above, and that it *is* equal to a fraction of integers $\frac{a}{b}$. We know that every fraction of integers may be written in lowest terms (i.e., so that a and b have no common factors), so we assume that this is the case for $\frac{a}{b}$.

2. Now, $\sqrt{2} = \frac{a}{b}$ implies that $2 = \frac{a^2}{b^2}$, whence $2b^2 = a^2$. This means that a^2 is even, in which case a itself is even (this is easy to check). Thus, $a = 2n$ for some integer n, and so $2b^2 = (2n)^2 = 4n^2$, in which case $b^2 = 2n^2$. By the same reasoning, we conclude that b itself is even. But this contradicts our assumption that the fraction $\frac{a}{b}$ was in lowest terms.

We conclude that $\sqrt{2}$ *cannot* be equal to a fraction of integers.