Counting Square-Free Integers

Problem 1. Let $p > 1$. By comparing the sum with an integral as in class, show that for any $x > 1$

$$\sum_{n>x} \frac{1}{n^p} = O(x^{1-p}).$$

In Friday's lecture we used this fact with $p = 2$.

Problem 2. Let n be a positive integer. From previous homework, we know that we can write $n = qm^2$ where q is either 1 or square-free. Let d be a positive integer. Show that $d^2|n$ if and only if $d|m$.

Problem 3. If μ denotes the Möbius function, then

$$\mu^2(n) = \begin{cases} 1 & \text{if } n \text{ is squarefree or 1,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, μ^2 detects square-free integers.

a. If $n = qm^2$ with q square-free, as in the previous problem, show that $\mu^2(n) = \delta(m)$.

b. Use Problem 2 and the fact that $\delta = \mathbf{1} * \mu$ to show that

$$\mu^2(n) = \sum_{d^2|n} \mu(d).$$

Problem 4. Let $Q(x)$ denote the number of square-free integers less than or equal to x.

a. Show that $Q(x)$ is the summatory function of μ^2. That is, show that

$$Q(x) = \sum_{n \leq x} \mu^2(n).$$

[Note: This *is not* the book's definition of a summatory function.]

b. Use Problem 3b and the method used in class to show that

$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$