Math 25 Problems

October 22, 2004

Recall the formula for the n^{th} prime number proven in class:

$$p_n = 1 + \sum_{m=1}^{2^n} \chi_n \left(-1 + \sum_{k=1}^{m} F(k) \right)$$

where

$$F(k) = \left[\cos^2 \left(\pi \frac{(k-1)! + 1}{k} \right) \right]$$

and

$$\chi_n(k) = \begin{cases} 1 & \text{if } k < n \\ 0 & \text{if } k \geq n \end{cases} .$$

Problem 1. Show that we can take

$$\chi_n(a) = \left[\sqrt[n]{\frac{n}{1+a}} \right].$$

That is, show that

$$1 \leq \sqrt[n]{\frac{n}{1+a}} < 2$$

if $1 \leq a < n$ and

$$0 \leq \sqrt[n]{\frac{n}{1+a}} < 1$$

if $a \geq n$.

Problem 2. The formula we've given for p_n is useless for all practical purposes. The sums involved get very long very quickly, preventing us from actually evaluating them in any reasonable amount of time.

a. Suppose we want to compute the prime 3 from the formula. How many terms does the outer sum in the formula have?

b. Suppose we want to compute the prime 13 from the formula. How many terms does the outer sum in the formula have?

c. Suppose we want to compute the prime 97 from the formula. How many terms does the outer sum in the formula have?

Problem 3. In addition to the ridiculous size of the outer sum, keep in mind the factorials that need to be computed as well. Using the result of the previous exercise:

a. What is the largest factorial occurring in the formula needed to compute the prime 97?

b. Show that the decimal expansion of the factorial from part (a) terminates with over 8 million zeros!

One problem with the formula we've given stems from the fact that we needed a generic upper bound for p_n in order to compute p_n. We used the bound $p_n \leq 2^n$, which is not very good in the sense that p_n is *much* smaller than 2^n for large n: the prime number theorem implies that, in fact, $p_n \sim n \log n$. Even if we were to replace 2^n by something substantially smaller, the factorials occurring in the formula would still be too large to actually compute. So it seems that the formula we've given is no more than a mathematical curiosity.