Extra Problem Set 1

Problem 1. Let $a \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ and $g(x) = (x+a)^n$. Using only the formal definition of the derivative of a polynomial and the binomial theorem, prove that

$$g'(x) = n(x+a)^{n-1}.$$

You *may not* use results from calculus.

Problem 2. Use the result of Problem 1 to show that for $k \leq n$

$$g^{(k)}(x) = n(n-1)(n-2)\cdots(n-(k-1))(x-a)^{n-k}$$

where, as before, $g(x) = (x+a)$.

Problem 3. Let $f(x)$ be a polynomial with integer coefficients. Let h be the number of solutions to $f(x) \equiv 0 \pmod{p^2}$ and suppose that $p \nmid h$. Show that $f(x) \equiv 0 \pmod{p^k}$ has a solution for all positive integers k.

Problem 4. Show that $x^2 + 1 \equiv 0 \pmod{5^k}$ has solutions for every $k \geq 1$. [Since $x^2 + 1 = 0$ has no integer (or even real) solutions, this shows that the existence of solutions mod p^k for every k does not imply the existence of integer solutions.]