Some Logic and Set Theory

1. First-Order Logic

Why use logical symbols? Brevity and clarity: $x < y$ is much shorter than "x is less than y", and the savings grows as the statement becomes more complicated. Expressing a mathematical statement symbolically can also make it more obvious what needs to be done with it, and however carefully words are used they may admit some ambiguity.

The simplest formula is a single symbol (or assertion) which can be either true or false. There are several ways to modify formulas.

The *conjunction* of formulas φ and ψ is written "φ and ψ", "$\varphi \wedge \psi$", or "φ & ψ". It is true when both φ and ψ are true, and false otherwise. Logically "and" and "but" are equivalent, and so are φ & ψ and ψ & φ, though in natural language there are some differences in connotation.

The *disjunction* of φ and ψ is written "φ or ψ", or "$\varphi \vee \psi$". It is false when both φ and ψ are false, and true otherwise. That is, $\varphi \vee \psi$ is true when *at least one* of φ and ψ is true; it is *inclusive or*. Natural language tends to use *exclusive or*, where only one of the clauses will be true, though there are exceptions. One such: "Would you like sugar or cream in your coffee?" Again, $\varphi \vee \psi$ and $\psi \vee \varphi$ are equivalent.

The *negation* of φ is written "not(φ)", "not-φ", "$\neg\varphi$", or "$\sim\varphi$". It is true when φ is false and false when φ is true. The potential difference from natural language negation is that $\neg\varphi$ must cover **all** cases where φ fails to hold, and in natural language the scope of a negation is sometimes more limited. Note that $\neg\neg\varphi = \varphi$.

How does negation interact with conjunction and disjunction? φ & ψ is false when φ, ψ, or both are false, and hence its negation is $(\neg\varphi) \vee (\neg\psi)$. $\varphi \vee \psi$ is false only when both φ and ψ are false, and so its negation is $(\neg\varphi)\&(\neg\psi)$. We might note in the latter case that this matches up with natural language's "neither...nor" construction. These two negation rules are called *DeMorgan's Laws*.

Exercise 1.1. Simplify the following formulas.

(i) $\varphi \;\&\; ((\neg\varphi) \vee \psi)$.
(ii) $(\varphi \;\&\; (\neg\psi) \;\&\; \theta) \vee (\varphi \;\&\; (\neg\psi) \;\&\; (\neg\theta))$.
(iii) $\neg((\varphi \;\&\; \neg\psi) \;\&\; \varphi)$.

There are two classes of special formulas to highlight now. A *tautology* is always true; the classic example is $\varphi \vee (\neg\varphi)$ for any formula φ. A *contradiction* is always false; here the example is φ & $(\neg\varphi)$.

To say φ *implies* ψ ($\varphi \to \psi$ or $\varphi \Rightarrow \psi$) means whenever φ is true, so is ψ. We call φ the *antecedent* and ψ the *consequent* of the implication. We also say φ is *sufficient* for ψ (since whenever we have φ we have ψ, though we may also have ψ when φ is false), and ψ is *necessary* for φ (since it is impossible to have φ without ψ). Clearly $\varphi \to \psi$ should be true when both formulas are true, and it should be false if φ is true but ψ is false. It is maybe not so clear what to do when φ is false; this is clarified by rephrasing implication as disjunction (which is often how it is defined in the first place). $\varphi \to \psi$ means either ψ holds or φ fails; i.e., $\psi \vee (\neg\varphi)$. The truth of that statement lines up with our assertions earlier, and gives truth values for when φ is false – namely, that the implication is true. Another

Rebecca Weber, 2012

way to look at this is to say $\varphi \to \psi$ is only false when proven false, and that can only happen when you see a true antecedent and a false consequent. From this it is clear that $\neg(\varphi \to \psi)$ is $\varphi \mathbin{\&} (\neg\psi)$.

There are a number of reformulations or related statements to the implication $\varphi \to \psi$. Its *converse* is $\psi \to \varphi$; a statement and its converse may have different truth values. Its *contrapositive* is $\neg\psi \to \neg\varphi$, which is logically equivalent to the original statement. Therefore when proving an implication you can either assume φ is true and deduce ψ, or assume ψ is false (assume $\neg\psi$ is true) and show φ is also false (deduce $\neg\varphi$).

There is an enormous difference between implication in natural language and implication in logic. Implication in natural language tends to connote causation, whereas the truth of $\varphi \to \psi$ need not give any connection at all between the meanings of φ and ψ. It could be that φ is a contradiction, or that ψ is a tautology. Also, in natural language we tend to dismiss implications as irrelevant or meaningless when the antecedent is false, whereas to have a full and consistent logical theory we cannot throw those cases out.

Example 1.2. The following are true implications:

- If fish live in the water, then earthworms live in the soil.
- If rabbits are aquamarine blue, then earthworms live in the soil.
- If rabbits are aquamarine blue, then birds drive cars.

The negation of the final statement is "Rabbits are aquamarine blue but birds do not drive cars." Its converse is "If birds drive cars, then rabbits are aquamarine blue." Its contrapositive is "If birds do not drive cars, then rabbits are not aquamarine blue."

The statement "If fish live in the water, then birds drive cars" is an example of a false implication.

Equivalence is two-way implication and indicated by a double-headed arrow: $\varphi \leftrightarrow \psi$ or $\varphi \Leftrightarrow \psi$. It is an abbreviation for $(\varphi \to \psi) \mathbin{\&} (\psi \to \varphi)$, and is true when φ and ψ are either both true or both false. Verbally we might say "φ if and only if ψ", which is often abbreviated to "φ iff ψ". In terms of just conjunction, disjunction, and negation, we may write equivalence as $(\varphi \mathbin{\&} \psi) \vee ((\neg\varphi) \mathbin{\&} (\neg\psi))$. Its negation is *exclusive or*, $(\varphi \vee \psi) \mathbin{\&} \neg(\varphi \mathbin{\&} \psi)$. Typically to prove an equivalence you prove implications in each direction, replacing either or both with its contrapositive. For example, $\varphi \leftrightarrow \psi$ may be proved as $(\varphi \to \psi) \mathbin{\&} (\psi \to \varphi)$ or $(\neg\psi \to \neg\varphi) \mathbin{\&} (\psi \to \varphi)$, or other combinations.

Exercise 1.3. Negate the following statements.

(i) 56894323 is a prime number.
(ii) If there is no coffee, I drink tea.
(iii) John watches but does not play.
(iv) I will buy the blue shirt or the green one.

Exercise 1.4. The following are statements related to the equivalence "Fish are talkative if and only if oranges have teeth." Determine all pairs of statements whose proof would be a proof of the original statement.

(i) If fish are talkative, then oranges have teeth.
(ii) If fish are not talkative, then oranges have teeth.
(iii) If fish are talkative, then oranges don't have teeth.
(iv) If fish are not talkative, then oranges don't have teeth.

(v) If oranges have teeth, then fish are talkative.
(vi) If oranges don't have teeth, then fish are talkative.
(vii) If oranges have teeth, then fish are not talkative.
(viii) If oranges don't have teeth, then fish are not talkative.

If we stop here, we have *propositional* (or *sentential*) logic. These formulas usually look something like $[A \vee (B \& C)] \to C$ and their truth or falsehood depends on the truth or falsehood of the assertions A, B, and C. We continue on to *predicate* logic, which replaces these assertions with statements such as $(x < 0)$ & $(x + 100 > 0)$, which will be true or false depending on the value substituted for the variable x. We will be able to turn those formulas into statements which are true or false inherently via *quantifiers.* Note that writing $\varphi(x)$ indicates the variable x appears in the formula φ.

The *existential* quantification $\exists x$ is read "there exists x." The formula $\exists x \varphi(x)$ is true if for some value n the unquantified formula $\varphi(n)$ is true. *Universal* quantification, on the other hand, is $\forall x \varphi(x)$ ("for all x, $\varphi(x)$ holds"), true when no matter what n we fill in for x, $\varphi(n)$ is true.

Quantifiers must have a specified set of values to range over, because the truth value of a formula may be different depending on this *domain of quantification.* For example, take the formula

$$(\forall x)(x \neq 0 \to (\exists y)(xy = 1)).$$

This asserts every nonzero x has a multiplicative inverse. If we are letting our quantifiers range over the real numbers or the rational numbers, this statement is true, because the reciprocal of x is available to play the role of y. However, in the integers or natural numbers this is false, because $1/x$ is only in the domain when x is ± 1.

When proving a statement of the form "there exists x," it suffices to produce an example of an x that works. When proving a statement of the form "for all x," you must take an arbitrary x and, using only properties common to all possible examples, prove that the rest of the statement is true.

Exercise 1.5. Consider the natural numbers, integers, rational numbers, and real numbers. Over which domains of quantification are each of the following statements true?

(i) $(\forall x)(x \geq 0)$.
(ii) $(\exists x)(5 < x < 6)$.
(iii) $(\forall x)((x^2 = 2) \to (x = 5))$.
(iv) $(\exists x)(x^2 - 1 = 0)$.
(v) $(\exists x)(x^3 + 8 = 0)$.
(vi) $(\exists x)(x^2 - 2 = 0)$.

When working with multiple quantifiers the order of quantification can matter a great deal. For example, take the two formulas

$$\varphi = (\forall x)(\exists y)(x \cdot x = y);$$

$$\psi = (\exists y)(\forall x)(x \cdot x = y).$$

φ says "every number has a square" and is true in our typical domains. However, ψ says "there is a number which is all other numbers' square" and is true only if you are working over the domain containing only 0 or only 1.

Exercise 1.6. Over the real numbers, which of the following statements are true? Over the natural numbers?

(i) $(\forall x)(\exists y)(x + y = 0)$.
(ii) $(\exists y)(\forall x)(x + y = 0)$.
(iii) $(\forall x)(\exists y)(x \leq y)$.
(iv) $(\exists y)(\forall x)(x \leq y)$.
(v) $(\exists x)(\forall y)(x < y^2)$.
(vi) $(\forall y)(\exists x)(x < y^2)$.
(vii) $(\forall x)(\exists y)(x \neq y \to x < y)$.
(viii) $(\exists y)(\forall x)(x \neq y \to x < y)$.

The order of operations when combining quantification with conjunction or disjunction can also make the difference between truth and falsehood.

Exercise 1.7. Over the real numbers, which of the following statements are true? Over the natural numbers?

(i) $(\forall x)(x \geq 0 \vee x \leq 0)$.
(ii) $(\forall x)(x \geq 0) \vee (\forall x)(x \leq 0)$.
(iii) $(\exists x)(x \leq 0 \;\&\; x \geq 5)$.
(iv) $(\exists x)(x \leq 0) \;\&\; (\exists x)(x \geq 5)$.

How does negation work for quantifiers? If $\exists x \varphi(x)$ fails, it means no matter what value we fill in for x the formula obtained is false – i.e., $\neg(\exists x \varphi(x)) \leftrightarrow \forall x(\neg\varphi(x))$. Likewise, $\neg(\forall x \varphi(x)) \leftrightarrow \exists x(\neg\varphi(x))$: if φ does not hold for all values of x, there must be an example for which it fails. If we have multiple quantifiers, the negation walks in one by one, flipping each quantifier and finally negating the predicate inside. For example:

$$\neg[(\exists x)(\forall y)(\forall z)(\exists w)\varphi(x, y, z, w)] \leftrightarrow (\forall x)(\exists y)(\exists z)(\forall w)(\neg\varphi(x, y, z, w)).$$

Exercise 1.8. Negate the following sentences.

(i) $(\forall x)(\exists y)(\forall z)((z < y) \to (z < x))$.
(ii) $(\exists x)(\forall y)(\exists z)(xz = y)$.
(iii) $(\forall x)(\forall y)(\forall z)(y = x \;\vee\; z = x \;\vee\; y = z)$.
(bonus: over what domains of quantification would this be true?)

2. Sets

A *set* is a collection of objects. If x is an element of a set A, we write $x \in A$, and otherwise $x \notin A$. The size of a set A, $|A|$, is the number of elements it contains (in the finite case; otherwise, for our purposes, it is infinite). Two sets are *equal* if they have the same elements; if they have no elements in common they are called *disjoint*. The set A is a *subset* of a set B if all of the elements of A are also elements of B; this is denoted $A \subseteq B$. If we know that A is not equal to B, we may write $A \subset B$ or (to emphasize the non-equality) $A \subsetneq B$.

We may write a set using an explicit list of its elements, such as {red, blue, green} or $\{5, 10, 15, \ldots\}$. When writing down sets, order does not matter and repetitions do not count. That is, $\{1, 2, 3\}$, $\{2, 3, 1\}$, and $\{1, 1, 2, 2, 3, 3\}$ are all representations of the same set. We may also write it in notation that may be familiar to you from calculus:

$$A = \{x : (\exists y)(y^2 = x)\}$$

This is the set of all values we can fill in for x that make the logical predicate $(\exists y)(y^2 = x)$ true. We are always working within some fixed *universe*, a set which contains all of our sets. The domain of quantification is all elements of the universe, and hence the contents of the set above will vary depending on what our universe is. If we are living in the integers it is the set of perfect squares; if we are living in the real numbers it is the set of all non-negative numbers.

Given two sets, we may obtain a third from them in several ways. First there is *union*: $A \cup B$ is the set containing all elements that appear in at least one of A and B. Next *intersection*: $A \cap B$ is the set containing all elements that appear in both A and B. We can subtract: $A - B$ contains all elements of A that are *not* also elements of B. You will often see $A \setminus B$ for set subtraction, but we will use ordinary minus because the slanted minus is sometimes given a different meaning in computability theory. Finally, we can take their *Cartesian product*: $A \times B$ consists of all ordered pairs that have their first entry an element of A and their second an element of B. We may take the product of more than two sets to get ordered triples, quadruples, quintuples, and in general *n-tuples*. If we take the Cartesian product of n copies of A, we may abbreviate $A \times A \times \ldots \times A$ as A^n. A generic ordered n-tuple from A^n will be written $(x_1, x_2, \ldots, x_n)$, where x_i are all elements of A.

Example 2.1. Let $A = \{x, y\}$ and $B = \{y, z\}$. Then $A \cup B = \{x, y, z\}$, $A \cap B = \{y\}$, $A - B = \{x\}$, $B - A = \{z\}$, and $A \times B = \{(x, y), (x, z), (y, y), (y, z)\}$.

Some sets have standard symbols to represent them. $\emptyset$ is the *empty set*, the set with no elements. $\mathbb{N}$ is the *natural numbers*, the set $\{0, 1, 2, 3, \ldots\}$. $\mathbb{Z}$ is the integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers.

When a universe U is fixed we can define *complement*. The complement of A, denoted $\overline{A}$, is all the elements of the universe that are not in A; i.e., $\overline{A} = U - A$.

Exercise 2.2. Convert the list or description of each of the following sets into notation using a logical predicate. Assume the domain of quantification is $\mathbb{N}$.

(i) $\{2, 4, 6, 8, 10, \ldots\}$.
(ii) $\{4, 5, 6, 7, 8\}$.
(iii) The set of numbers that are cubes.
(iv) The set of pairs of numbers such that one is twice the other (in either order).
(v) The intersection of the set of square numbers and the set of numbers that are divisible by 3.
(vi) [For this and the next two, you'll need to use $\in$ in your logical predicate.] $A \cup B$ for sets A and B.
(vii) $A \cap B$ for sets A and B.
(viii) $A - B$ for sets A and B.

We can also take unions and intersections of infinitely many sets. If we have sets A_i for $i \in \mathbb{N}$, these are

$$\bigcup_i A_i = \{x : (\exists i)(x \in A_i)\}$$

$$\bigcap_i A_i = \{x : (\forall i)(x \in A_i)\}.$$

The i under the union or intersection symbol is also sometimes written "$i \in \mathbb{N}$".

Exercise 2.3. For $i \in \mathbb{N}$, let $A_i = \{0, 1, \ldots, i\}$ and let $B_i = \{0, i\}$. What are $\bigcup_i A_i$, $\bigcup_i B_i$, $\bigcap_i A_i$, and $\bigcap_i B_i$?

If two sets are given by descriptions instead of explicit lists, we must prove one set is a subset of another by taking an arbitrary element of the first set and showing it is also a member of the second set. For example, to show the set of people eligible for President of the United States is a subset of the set of people over 30, we might say: Consider a person in the first set. That person must meet the criteria listed in the US Constitution, which includes being at least 35 years of age. Since 35 is more than 30, the person we chose is a member of the second set.

We can further show that this containment is proper, by demonstrating a member of the second set who is not a member of the first set. For example, a 40-year-old who is not a U.S. citizen. Note that $A \subseteq B$ is a universally quantified property: every element of A is an element of B. Proper containment, or nonequality generally, is an existentially quantified property: there exists an element of B that is not an element of A.

Exercise 2.4. Prove that the set of squares of even numbers, $\{x : \exists y(x = (2y)^2)\}$, is a proper subset of the set of multiples of 4, $\{x : \exists y(x = 4y)\}$.

To prove two sets are equal, there are three options: show the criteria for membership on each side are the same, manipulate set operations until the expressions are the same, or show each side is a subset of the other side.

An extremely basic example of the first option is showing $\{x : \frac{x}{2}, \frac{x}{4} \in \mathbb{N}\} = \{x : (\exists y)(x = 4y)\}$. For the second, we have a bunch of *set identities*, things like de Morgan's Laws,

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B},$$

and distribution laws,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

To prove identities we have to turn to the first or third option.

Example 2.5. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

We work by showing each set is a subset of the other. Suppose first that $x \in A \cup (B \cap C)$. By definition of union, x must be in A or in $B \cap C$. If $x \in A$, then x is in both $A \cup B$ and $A \cup C$, and hence in their intersection. On the other hand, if $x \in B \cap C$, then x is in both B and C, and hence again in both $A \cup B$ and $A \cup C$.

Now suppose $x \in (A \cup B) \cap (A \cup C)$. Then x is in both unions, $A \cup B$ and $A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If, however, $x \notin A$, then x must be in both B and C, and therefore in $B \cap C$. Again, we obtain $x \in A \cup (B \cap C)$.

Notice that in the $\subseteq$ direction we used two cases that could overlap, and did not worry whether we were in the overlap or not. In the $\supseteq$ direction, we could only assert $x \in B$ *and* $x \in C$ if we knew $x \notin A$ (although it is certainly possible for x to be in all three sets), so forbidding the first case was part of the second case.

Exercise 2.6. Using any of the three options listed above, as long as it is applicable, do the following.

(i) Prove intersection distributes over union (i.e., for all A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$).
(ii) Prove de Morgan's Laws.
(iii) Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ for any sets A and B.

3. Proofs

Here are some proofs to try that don't involve linear algebra. For rigorous definitions: An integer is even if it is $2n$ for some integer n, and odd if it is $2n+1$ for some integer n. For integers, "divisible" means "evenly divisible;" that is, k divides ℓ if there is some integer n such that $kn = \ell$. A positive integer is prime if it is greater than 1 and it is only divisible by itself and 1. A real number is rational if it equals a fraction of integers, and a fraction is in least terms if its numerator and denominator have no common divisors besides 1.

(1) The integer n is even if and only if $n+1$ is odd.
(2) For integers a, b, c, if a divides b and a divides c, then a divides $b - c$.
(3) For integers a, b, c, if ac divides bc then a divides b.
(4) $\neg(\forall m)(\forall n)(3m + 5n = 12)$ (quantifiers range over $\mathbb{N}$)
(5) For any integer n, the number $n^2 + n + 1$ is odd.
(6) If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.
(7) For nonempty sets A and B, $A \times B = B \times A$ if and only if $A = B$.
(8) $\sqrt{2}$ is irrational. (Hint: work by contradiction, assuming there is a fraction of integers in least terms that equals $\sqrt{2}$.)
(9) For r any nonzero real number, there is an integer n such that $|1/n| < |r|$.
(10) Give a direct proof that the product of two odd integers must be odd, and then give a proof by the contrapositive.
(11) A *prime triple* is a set of three consecutive odd integers that are all prime. Prove that $3, 5, 7$ is the only prime triple.