What do we know?

Since the first quiz. . . that is, essentially Chapter 8.

Linear transformation: function between vector spaces that respects vector addition and scalar multiplication. That is, $T : V \to W$ such that $\forall \boldsymbol{X}, \boldsymbol{Y} \in V, \forall r \in \mathbb{R}$, $T(\boldsymbol{X} + \boldsymbol{Y}) = T(\boldsymbol{X}) + T(\boldsymbol{Y})$ and $T(r\boldsymbol{X}) = rT(\boldsymbol{X})$.

Extends to any linear combination of vectors of V.

General examples: zero transformation maps all vectors of V to W's zero vector; identity transformation, for $V = W$, takes every vector to itself.

For $E \subseteq V$, $T(E)$ is the set of all images of vectors in E. $T(\mathcal{L}(E)) = \mathcal{L}(T(E))$; clearly this is a subspace of W. $\mathrm{Im}(T) = T(V)$, the image of V under T, is an example of such a subspace.

$\ker(T) = \{\boldsymbol{X} \in V : T(\boldsymbol{X}) = \boldsymbol{0}\}$, the kernel of T. It is a subspace of V.

For any $T : V \to W$, $\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(V)$.

The collection of all linear transformations between V and W, $\mathcal{L}(V, W)$, is a vector space under standard function addition and scalar multiplication. It is a subspace of the vector space $\mathrm{Fun}(V, W)$, which treats V as a set and includes maps that are not linear.

The composition of two linear transformations is also a linear transformation.

Given images in W for a basis of V, we may construct $T : V \to W$ by linear extension: every $\boldsymbol{X} \in V$ will have the form $a_1\boldsymbol{E}_1 + \cdots + a_n\boldsymbol{E}_n$ for the basis $\boldsymbol{E}_i$, so let $T(\boldsymbol{X}) = a_1T(\boldsymbol{E}_1) + \cdots + a_nT(\boldsymbol{E}_n)$.

Every linear transformation is determined by its action on any basis of V, and every possible set of images of basis vectors of V gives rise to a linear transformation.

T is injective if and only if its kernel is $\{\boldsymbol{0}\}$, which is if and only if the set of images under T of a basis of V is linearly independent in W.

T is surjective if and only if the set of images under T of a basis of V spans W.

If $\dim(V) < \dim(W)$, T cannot be surjective. If $\dim(V) > \dim(W)$, T cannot be injective. If $\dim(V) = \dim(W)$, T may be bijective or neither injective nor surjective, but not one without the other.

An isomorphism is a linear transformation that has an inverse: $T : V \to W$ such that there is $S : W \to V$ with $S \circ T$ and $T \circ S$ the identity transformations on V and W, respectively.

$T : V \to W$ is an isomorphism if and only if it is a bijection, which is if and only if $\dim(V) = \dim(W)$ and T is either injective or surjective.

If $\dim(V) = \dim(W)$, there exists an isomorphism between V and W.