Math 24
Winter 2010

Definition: For vector spaces V and W over a field F, we define a new vector space $V \oplus W$ by setting

$$V \oplus W = \{(v, w) \mid v \in V \ \&\ w \in W\}$$

and defining addition and scalar multiplication coordinatewise:

$$(v, w) + (v', w') = (v + w,\ v' + w');$$

$$a(v, w) = (av,\ aw).$$

Proposition: The structure $V \oplus W$ satisfies $VS3$. (There is an additive identity in $V \oplus W$.)

Proof: Let 0_V and 0_W be the additive identities of V and W respectively. In $V \oplus W$, define $0 = (0_V,\ 0_W)$. To show that 0 is an additive identity for $V \oplus W$, let (v, w) be any element of $V \oplus W$. Then

$$\begin{aligned}(v, w) + 0 &= (v, w) + (0_V, 0_W) && \text{(by the definition of 0)}\\ &= (v + 0_V,\ w + 0_W) && \text{(by the definition of addition in } V \oplus W\text{)}\\ &= (v, w) && \text{(because } 0_V \text{ and } 0_W \text{ are additive identities).}\end{aligned}$$

Another way of phrasing this proof:

Proof: Because V and W are vector spaces, in V we have $v + 0 = v$, and in W we have $w + 0 = w$. Using this and the definition of addition in $V \oplus W$, we see

$$(v, w) + (0, 0) = (v + 0,\ w + 0) = (v, w).$$

This shows that $(0, 0)$ is an additive identity for $V \oplus W$.

Comments: The first proof states explicitly that (v, w) is an arbitrary element of $V \oplus W$, while the second proof assumes the reader will infer this. In contrast, the second proof explicitly explains that we are using the fact that V and W themselves satisfy the vector space axioms.

How much you state explicitly and how much you leave the reader to assume depends on context, on your audience, and on the complexity of your proof. In general, the more complicated the proof, the more important it is to state things explicitly (and clearly).

Assume your reader is another student in this class. So, for example, in this first proof using the vector space axioms, it is appropriate to point out that when you are using the vector space axioms. In later proofs, you may generally assume your reader knows the vector space axioms, and use them without comment.

Notice that both proofs use complete sentences and correct grammar, explain what they are proving, and use equations and page placement (including white space) to enhance clarity and readability.