Math 24
Winter 2010

Here is an example of a proof by induction. This proposition is also proved in the textbook as Theorem 1.9 on page 44.[1]

Proposition: If V is any vector space, and $S = \{x_1, x_2, \ldots, x_n\}$ is a finite subset of V that generates V, then there is a linearly independent subset of S that generates V.

Proof: By induction on n.

Base Step: If $n = 1$, then $S = \{x_1\}$.
If $x_1 = 0$, then $V = \mathit{span}(S) = \{0\}$, and $\emptyset$ is a linearly independent subset of S that generates S.
If $x_1 \neq 0$, then S is itself linearly independent. (Fact (2) on page 37 of the textbook.)

Inductive Step: Assume[2] that any set of n vectors that generates V has a linearly independent subset that generates V.
Suppose that $S = \{x_1, x_2, \ldots x_n, x_{n+1}\}$ generates V. We will show that S has a linearly independent subset $S' \subseteq S$ that generates V.
If S is itself linearly independent, then we can take $S' = S$.
If not, we can write 0 as a nontrivial linear combination of vectors in S,

$$a_1 x_1 + \cdots + a_n x_n + a_{n+1} x_{n+1} = 0,$$

where $a_i \neq 0$ for at least one i. Without loss of generality, we may assume $a_{n+1} \neq 0$. Therefore we can write

$$x_{n+1} = \frac{-a_1}{a_{n+1}} x_1 + \cdots + \frac{-a_n}{a_{n+1}} x_n.$$

Now we will show that $\{x_1, x_2, \ldots, x_n\}$ generates V. Let $x \in V$; we must show that $x \in \mathit{span}(\{x_1, x_2, \ldots, x_n\})$. Because S generates V, we can write

$$x = b_1 x_1 + \cdots b_n x_n + b_{n+1} x_{n+1} = b_1 x_1 + \cdots b_n x_n + b_{n+1} \left(\frac{-a_1}{a_{n+1}} x_1 + \cdots + \frac{-a_n}{a_{n+1}} x_n \right).$$

This shows that $\{x_1, x_2, \ldots, x_n\}$ generates V.
Finally, by the inductive hypothesis, $\{x_1, x_2, \ldots, x_n\}$ has a linearly independent subset S' that generates V. Since S' is also a subset of S, we are done.

Conclusion: For every natural number $n \geq 1$, if $S = \{x_1, x_2, \ldots, x_n\}$ generates V, then S has a linearly independent subset that generates V.

[1] The proof in the textbook is not given as a formal proof by induction, but there is an induction hidden in that proof. Can you see why?

[2] The assumption we make here is called the *inductive hypothesis*.