Math 24
Winter 2010
Quiz 8

1. $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, and the eigenvalues of A are 1 and 2.

(a) The eigenspace of A corresponding to eigenvalue $\lambda = 1$ consists of all solutions to the equation $B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ where B is the matrix:

$$B = A - \lambda I = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(b) If the reduced row echelon form of B is $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, give a basis for the eigenspace of A corresponding to eigenvalue $\lambda = 1$.

The matrix equation $B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ is equivalent to the matrix equation

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or to the single scalar equation $x_1 - x_2 + x_3 = 0$, or $x_1 = x_2 - x_3$. The complete solution to this is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and a basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

2. An eigenvector for the eigenvalue $\lambda = 2$ of the matrix A in problem (1) is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

(a) Give a matrix Q such that either QAQ^{-1} or $Q^{-1}AQ$ is a diagonal matrix D. (Be sure to say which of QAQ^{-1} or $Q^{-1}AQ$ is equal to D.)

$Q^{-1}AQ$ is diagonal, where Q is the change of coordinate matrix from a basis of eigenvectors to standard coordinates. The columns of Q are the elements of the basis of eigenvectors, so

$$Q = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(b) What is D?

D is the matrix of L_A in the basis of eigenvectors, a diagonal matrix whose diagonal elements are the eigenvalues (in the same order as the corresponding eigenvectors appear as columns of Q):

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

3. Define:

(a) The vector space V is the direct sum of subspaces W_1 and W_2.

V is the direct sum of W_1 and W_2 if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

(b) The linear operator $T : V \to V$ is diagonalizable.

T is diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a digaonal matrix.

Note: This is the definition. A number of people wrote something like, "The characteristic polynomial of T factors, and the dimension of each eigenspace equals the multiplicity of the corresponding eigenvalue." There is a theorem saying that T is diagonalizable if and only if this is the case, so I gave full credit for that answer — but it is not, in fact, the definition of diagonalizable.