NAME:________________________________

Math 24
Winter 2010
Quiz 5

1. Let α be the standard ordered basis for $\mathbb{R}^2$, and $\beta = \{(1,2),\ (2,-1)\}$.

   (a) Find the change of coordinate matrices Q_β^α that changes from β coordinates to α coordinates and Q_α^β that changes from α coordinates to β coordinates.

   $$Q_\beta^\alpha = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

   $$Q_\alpha^\beta = (Q_\beta^\alpha)^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}.$$

   (b) If $A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$, find the matrix $[L_A]_\beta$. You may write the answer as a product of other matrices; you do not have to multiply them out.

   $$[L_A]_\beta = [L_A]_\beta^\beta = Q_\alpha^\beta [L_A]_\alpha^\alpha Q_\beta^\alpha = Q_\alpha^\beta A Q_\beta^\alpha = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

2. Use elementary row operations to transform the matrix $A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$ into the identity matrix. Express A as a product of elementary matrices.

   One possible sequence of operations is:

   (a.) Add row 1 to row 2 to get $\begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix}$.

   (b.) Multiply row 2 by $\frac{1}{6}$ to get $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

   (c.) Add (-3) times row 2 to row 1 to get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   If the elementary matrices corresponding to these row operations are E_a, E_b, and E_c respectively, we have

   $$E_c(E_b(E_a A)) = I.$$

Therefore

$$A = E_a^{-1} E_b^{-1} E_c^{-1}$$

(which we can see by multiplying both sides of the previous equation on the left, first by E_c^{-1}, then by E_b^{-1}, then by E_a^{-1}). Using the fact that the inverse elementary matrix corresponds to the inverse elementary row operation, and can be obtained by applying that row operation to the identity matrix, we see that

$$A = E_a^{-1} E_b^{-1} E_c^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

3. (a) Give a careful and complete statement of the Dimension Theorem.

   If T is a linear transformation whose domain is a finite dimensional vector space V, then

   $$nullity(T) + rank(T) = dim(V).$$

   (b) Suppose that V is an n-dimensional vector space and W is an m-dimensional vector space. Complete these statements:

   i. A linear transformation $T : V \to W$ is one-to-one if and only if $nullity(T) = \underline{\ 0\ }$.
   (We have seen that T is one-to-one if and only if $N(T) = \{0\}$.)

   ii. A linear transformation $T : V \to W$ is one-to-one if and only if $rank(T) = \underline{\ \mathrm{n}\ }$.
   (By the Dimension Theorem, $nullity(T) = 0$ if and only if $rank(T) = n$.)