Math 24

Winter 2010

Quiz 4 Sample Solutions

1. This problem is about linear functions from $\mathbb{R}^3$ to $\mathbb{R}^3$.

   (a) If $T(x, y, z) = (x - y, y + z, 2z - x)$, what is the matrix representing T in the standard ordered basis?

   $T(1, 0, 0) = (1, 0, -1)$, $T(0, 1, 0) = (-1, 1, 0)$, and $T(0, 0, 1) = (0, 1, 2)$, so the matrix is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$$

   (b) If $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, what is $L_A(x, y, z)$?

$$L_A(x, y, z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x + 2y,\ x + y + 2z,\ y + z)$$

2. TRUE or FALSE?

   (a) The vector spaces $M_{2\times 2}(F)$ and $P_3(F)$ are isomorphic.

   TRUE. Vector spaces of the same dimension over the same field are isomorphic.

   (b) If V is an n-dimensional vector space, α and β are two different ordered bases for V, and $T : V \to V$ is linear, then $[T]_\beta^\alpha [T]_\alpha^\beta$ is the identity matrix.

   FALSE. $[T]_\beta^\alpha [T]_\alpha^\beta = [T^2]_\alpha$. This is the identity matrix only if T^2 is the identity transformation.

   (c) If $T : V \to W$ is linear, $dim(V) = n$, and $dim(W) = n$, then T is an isomorphism if and only if $rank(T) = n$.

   TRUE. If linear transformation between n-dimensional vector spaces is onto, then it is an isomorphism.

   (d) There are linear transformations $U : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ and $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ such that UT is the identity transformation on its domain.

   FALSE. The range of U can have dimension at most 2 (since the domain of U has dimension 2), so the range of the composition UT cannot have dimension 3.

(e) There are linear transformations $U : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ and $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ such that TU is the identity transformation on its domain.

TRUE. For example, if $U(a + bx) = a + bx + 0x^2$ and $T(a + bx + cx^2) = a + bx$, then $TU(a + bx) = U(a + bx + 0x^2) = a + bx$

3. Let $\alpha = \{1,\ x,\ x^2\}$ be an ordered basis for $P_2(\mathbb{R})$, and $\beta = \{(1,0,0),\ (1,1,0,\ (1,1,1)\}$ be an ordered basis for $\mathbb{R}^3$. Suppose $U : \mathbb{R}^3 \to P_2(\mathbb{R})$ is a linear transformation defined by $U(a,b,c) = a + b + cx^2$ and $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ is a linear transformation, with $[T]_\alpha^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) What is $[U]_\beta^\alpha$?

The columns are the α-coordinates of $U(1,0,0)$, $U(1,1,0)$, and $U(1,1,1)$. $U(1,0,0) = 1$, $U(1,1,0) = 2$, and $U(1,1,1) = 2 + x^2$, so

$$[U]_\beta^\alpha = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) What is $[UT]_\alpha$?

$$[UT]_\alpha = [UT]_\alpha^\alpha = [U]_\beta^\alpha [T]_\alpha^\beta = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(c) What is $T(3x^2 + 2x + 1)$?

$$[T(3x^2 + 2x + 1)]_\beta = [T]_\alpha^\beta [3x^2 + 2x + 1]_\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \text{ so}$$

$$T(3x^2 + 2x + 1) = 3(1,0,0) + 5(1,1,0) + 4(1,1,1) = (12, 9, 4).$$