Math 24
Winter 2010
Quiz 3 Sample Solutions

1. A linear function $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y) = (x - y, y - z, z - x)$.

   (a) Find a basis for $N(T)$.
   $$\{(1, 1, 1)\}.$$
   (Solving $T(x, y, z) = (0, 0, 0)$ yields the solution $x = y = z$. Thus $N(T)$ is the line $x = y = z$, which is one-dimensional, so any nonzero vector on the line forms a basis.)

   (b) Find a generating set for $R(T)$.
   $$\{(1, 0, -1), (-1, 1, 0), (0, -1, 1)\}.$$
   (Since the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ span the domain of T, their images generate the range.)

   (c) Find a basis for $R(T)$.
   $$\{(1, 0, -1), (-1, 1, 0)\}.$$
   (Since the domain of T has dimension 3 and $N(T)$ has dimension 1, the dimension theorem tells us $R(T)$ has dimension 2. This means any two linearly independent vectors in $R(T)$ form a basis.)

2. Complete the definitions:

   (a) A nonempty subset S of a vector space V is *linearly dependent* if:
   ... there is a nontrivial linear combination of vectors in S that equals 0.

   (b) A subset S of the vector space is a *basis* for V if:
   ... S is linearly independent and spans (or generates) V.

   (c) A vector space V has *dimension* equal to n if:
   ... V has a basis containing (exactly) n elements.

3. TRUE or FALSE?

   (a) If $X = \{A_1, A_2, A_3, A_4\}$ is a linearly independent subset of $M_{2\times 2}(F)$, then X must generate $M_{2\times 2}(F)$.
   TRUE. ($M_{2\times 2}(F)$ has dimension 4 , so any set of 4 linearly independent vectors is a basis.)

   (b) If L is any linearly independent subset of a finite-dimensional vector space V, then there is a basis for V that contains L.
   TRUE. (This is a theorem, or more properly a corollary to the replacement theorem.)

(c) $T(f) = \int_0^1 f(x)\,dx$ is a linear transformation from $P(\mathbb{R})$ to $\mathbb{R}$.

TRUE. (Integration preserves addition and scalar multiplication.)

(d) If $T : \mathbb{R}^3 \to P_2(\mathbb{R})$ is linear and onto, then it must be the case that $N(T) = \{0\}$.

TRUE. (Since the dimensions of $domain(T)$ and $R(T)$ are both 3, by the dimension theorem, $N(T)$ must have dimension 0.)

(e) If $T : V \to W$ is a linear transformation, and w is any element of W, then $\{v \in V \mid T(v) = w\}$ is a subspace of V.

FALSE. (Since a subspace must always contain 0, and $T(0_V) = 0_W$, the set $\{v \in V \mid T(v) = w\}$ cannot be a subspace if $w \neq 0_W$.)