Math 24
Winter 2010
Quiz 2 Sample Solutions

1. Determine whether the following subset of $P_2(\mathbb{Q})$ is linearly independent. (Show your work.)
$$\{-2x^2 + x - 1,\ 2x^2 + 2x - 4,\ x + 1\}.$$

The set is linearly independent if and only if the only solution to
$a(-2x^2 + x - 1) + b(2x^2 + 2x - 4) + c(x + 1) = 0$
is $a = b = c = 0$. Solving the equation by setting the coefficients all equal to zero, we get
$$-2a + 2b = 0$$
$$a + 2b + c = 0$$
$$-a - 4b + c = 0$$
and solving this (for which you should show your work) does indeed yield $a = b = c = 0$ as the only solution.

2. A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *skew symmetric* if $b = -c$. Give a subset of $M_{2\times 2}(\mathbb{R})$ that generates the subspace of skew symmetric 2×2 matrices.
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$
The matrices in this set are all skew symmetric, and they span the entire subspace, as any skew symmetric matrix can be written
$\begin{pmatrix} a & -c \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
(Also, the set is clearly linearly independent, so it is in fact a basis.)

3. Complete these statements correctly.

   (a) A subset S of a vector space V is a basis for V if and only if
   ... S is linearly independent and spans V.
   or ... any vector in V can be written uniquely as a linear combination of vectors in S.

(b) If L is a linearly independent set of vectors and v is a vector, then $L \cup \{v\}$ is linearly independent if and only if

...either $v \notin span(L)$ or $v \in L$. (This last possibility is a subtle point. If $v \in L$, then $L \cup \{v\}$ is just L, as we haven't added anything new, and L is linearly independent.)

(c) Any intersection of subspaces of a vector space V is also a subspace of V.

(d) The vector space $M_{2\times 2}(F)$ can have subspaces of the following dimensions:

0, 1, 2, 3, and 4.

(e) If $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ and $S = \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -2 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$, we can see that A is not in the span of S because

...the matrices in S are all symmetric, and therefore every matrix in their span is symmetric, but A is not symmetric. (You can also show this by solving an appropriate system of linear equations.)