

Math 24
Winter 2010
Sample Solutions to the Midterm

(1.) (a.) Find a basis $\{v_1, v_2\}$ for the plane P in \mathbb{R}^3 with equation $3x + 2y - z = 0$.

We can take any two non-collinear vectors in the plane, for instance $v_1 = (1, 0, 3)$ and $v_2 = (0, 1, 2)$.

(b.) You know from multivariable calculus that the vector $v_3 = (3, 2, -1)$ is perpendicular to the plane P . Therefore $\beta = \{v_1, v_2, v_3\}$ is linearly independent, and forms an ordered basis for \mathbb{R}^3 .

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the perpendicular projection onto the plane P . In other words, $T(v)$ is the perpendicular projection of v onto P .

What is $[T]_\beta$?

As v_1 and v_2 are in the plane, $T(v_1) = v_1$ and $T(v_2) = v_2$. As v_3 is a vector perpendicular to the plane, its projection is the origin, $T(v_3) = 0$. Therefore

$$[T(v_1)]_\beta = [v_1]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, [T(v_2)]_\beta = [v_2]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, [T(v_3)]_\beta = [0]_\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c.) Let α be the standard ordered basis for \mathbb{R}^3 . Find the change of coordinate matrices Q_α^β that changes α coordinates into β coordinates, and Q_β^α that changes β coordinates into α coordinates.

Do not use matrix inversion (if you know how to invert matrices) to do this problem. Find each matrix by explicitly computing the coordinates of the appropriate vectors in the appropriate bases. If you wish, you can check your work by verifying that $Q_\alpha^\beta Q_\beta^\alpha = I$.

$Q_\beta^\alpha = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix}$, the matrix whose columns are the α (standard) coordinates of the vectors in β .

To find the β coordinates of the vectors in α (the standard basis vectors) we need to solve the vector equations:

$$(1, 0, 0) = a(1, 0, 3) + b(0, 1, 2) + c(3, 2, -1)$$

$$(0, 1, 0) = a(1, 0, 3) + b(0, 1, 2) + c(3, 2, -1)$$

$$(0, 0, 1) = a(1, 0, 3) + b(0, 1, 2) + c(3, 2, -1)$$

When we do this, we get

$$(1, 0, 0) = \frac{5}{14}(1, 0, 3) - \frac{6}{14}(0, 1, 2) + \frac{3}{14}(3, 2, -1)$$

$$(0, 1, 0) = -\frac{6}{14}(1, 0, 3) + \frac{10}{14}(0, 1, 2) + \frac{2}{14}(3, 2, -1)$$

that is, $(0, 0, 1) = \frac{3}{14}(1, 0, 3) + \frac{2}{14}(0, 1, 2) - \frac{1}{14}(3, 2, -1)$,

$$[(1, 0, 0)]_\beta = \begin{pmatrix} \frac{5}{14} \\ -\frac{6}{14} \\ \frac{3}{14} \end{pmatrix} \quad [(0, 1, 0)]_\beta = \begin{pmatrix} -\frac{6}{14} \\ \frac{10}{14} \\ \frac{2}{14} \end{pmatrix} \quad [(0, 0, 1)]_\beta = \begin{pmatrix} \frac{3}{14} \\ \frac{2}{14} \\ -\frac{1}{14} \end{pmatrix}.$$

Now, using these coordinates as the columns of Q_α^β , we have

$$Q_\alpha^\beta = \begin{pmatrix} \frac{5}{14} & -\frac{6}{14} & \frac{3}{14} \\ -\frac{6}{14} & \frac{10}{14} & \frac{2}{14} \\ \frac{3}{14} & \frac{2}{14} & -\frac{1}{14} \end{pmatrix}.$$

(d.) Find the matrix of T in the standard basis, $[T]_\alpha$.

If you want to use the result of part (b) but were not able to do part (b), you may pretend

$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This is not the correct answer to part (b).

$$[T]_\alpha = [T]_\alpha^\alpha = Q_\beta^\alpha [T]_\beta^\beta Q_\alpha^\beta = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{14} & -\frac{6}{14} & \frac{3}{14} \\ -\frac{6}{14} & \frac{10}{14} & \frac{2}{14} \\ \frac{3}{14} & \frac{2}{14} & -\frac{1}{14} \end{pmatrix}.$$

$$[T]_\alpha = \begin{pmatrix} \frac{5}{14} & -\frac{6}{14} & \frac{3}{14} \\ -\frac{6}{14} & \frac{10}{14} & \frac{2}{14} \\ \frac{3}{14} & \frac{2}{14} & \frac{13}{14} \end{pmatrix}.$$

(e.) Use your answers to find the perpendicular projection of $(2, 1, 1)$ onto the plane P .

If you want to check your work here, note that this point should in fact be on the plane P , and the line between it and $(2, 1, 1)$ should be perpendicular to P . If you used the wrong answer to part (b) supplied in part (d), the point you found will not be on P , but the line between it and $(2, 1, 1)$ will still be perpendicular to P .

$$[T(2, 1, 1)]_\alpha = [T]_\alpha [(2, 1, 1)]_\alpha = \begin{pmatrix} \frac{5}{14} & -\frac{6}{14} & \frac{3}{14} \\ -\frac{6}{14} & \frac{10}{14} & \frac{2}{14} \\ \frac{3}{14} & \frac{2}{14} & \frac{13}{14} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}.$$

$$T(2, 1, 1) = \left(\frac{1}{2}, 0, \frac{3}{2}\right).$$

(2.) Recall that if A is an $n \times m$ matrix, the linear transformation $L_A : F^m \rightarrow F^n$ is defined by $L_A(v) = Av$, and A is the matrix that represents L_A in the standard ordered bases for F^m and F^n .

This means that the columns of A are $L_A(e_1), L_A(e_2), \dots, L_A(e_m)$, written as column vectors, where e_1, e_2, \dots, e_m are the standard basis vectors for F^m .

(a.) Show that an $n \times n$ matrix is invertible if and only if its columns are linearly independent.

By Corollary 2 on page 108, an $n \times n$ matrix A is invertible if and only if the linear transformation $L_A : F^n \rightarrow F^n$ is invertible.

By Theorem 2.5 on page 71, L_A is invertible if and only if it is onto.

By Theorem 2.2 on page 68, the range of L_A is spanned by $L_A(e_1), L_A(e_2), \dots, L_A(e_n)$, so L_A is invertible if and only if the vectors $L_A(e_1), L_A(e_2), \dots, L_A(e_n)$ span F^n .

Since F^n is an n -dimensional vector space, by Corollary 2 on pages 47-48 the n vectors $L_A(e_1), L_A(e_2), \dots, L_A(e_n)$ span F^n if and only if they are linearly independent.

The vectors $L_A(e_1), L_A(e_2), \dots, L_A(e_n)$ are the columns of A , hence they are linearly independent if and only if the columns of A are linearly independent.

Putting all this together, we see that an $n \times n$ matrix A is invertible if and only if its columns are linearly independent.

(b.) Determine whether the matrix $\begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & 2 & -2 & 0 \\ 2 & 1 & -2 & -1 \\ -3 & 1 & 1 & 1 \end{pmatrix}$ is invertible. (Hint: Check the sum of its columns.)

Because its columns sum to zero, they are not linearly independent, and therefore by part (a) the matrix is not invertible.

(c.) Determine whether the linear transformation $T : M_{2 \times 2}(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b + c - 3d) + (2b - 2c)x + (2a + b - 2c - d)x^2 + (-3a + b + c + d)x^3$$

is invertible.

If $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for $M_{2 \times 2}(\mathbb{R})$ and $\alpha = \{1, x, x^2, x^3, \}$ is a basis for $P_3(\mathbb{R})$, then

$[T]_{\beta}^{\alpha} = \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & 2 & -2 & 0 \\ 2 & 1 & -2 & -1 \\ -3 & 1 & 1 & 1 \end{pmatrix}$. By part (b) this matrix is not invertible, and therefore by Theorem 2.18 on page 101, T is not invertible.

(3.) (a.) Find a basis for the subspace W of \mathbb{R}^4 consisting of all solutions to the following system of equations.

$$\begin{aligned}w + x + 2y + 3z &= 0 \\2w + 2x - 4y - 2z &= 0 \\y + z &= 0 \\-w - x + y &= 0.\end{aligned}$$

By subtracting multiples of equations 1 and 2 from equations 3 and 4 we can convert this to the equivalent system

$$\begin{aligned}w + x + z &= 0 \\y + z &= 0,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}w &= -x - z \\y &= -z.\end{aligned}$$

This system is satisfied by any vector of the form

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x - z \\ x \\ -z \\ z \end{pmatrix} = x \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix},$$

which shows the solution set is spanned by the set $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$.

Since this set is clearly linearly independent, it forms a basis for the solution set.

(b.) Find a basis for the subspace Z of \mathbb{R}^4 spanned by the vectors $(1, 1, 2, 3)$, $(2, 2, -4, -2)$, $(0, 0, 1, 1)$, and $(-1, -1, 1, 0)$.

We know (Theorem 1.9 on page 44) that some subset of this set forms a basis. Using our algorithm for finding a linearly independent set with the same span, we can begin with the last two vectors, $(0, 0, 1, 1)$, and $(-1, -1, 1, 0)$. Since neither of these is a multiple of the other, $\{(0, 0, 1, 1), (-1, -1, 1, 0)\}$ is a linearly independent set.

Checking to see whether $(2, 2, -4, -2)$ is in the span of $\{(0, 0, 1, 1), (-1, -1, 1, 0)\}$, we see that $(2, 2, -4, -2) = (-2)(0, 0, 1, 1) + (-2)(-1, -1, 1, 0)$, so $(2, 2, -4, -2)$ is in the span of $\{(0, 0, 1, 1), (-1, -1, 1, 0)\}$.

Now, checking to see whether $(1, 1, 2, 3)$ is in the span of $\{(0, 0, 1, 1), (-1, -1, 1, 0)\}$, we see that $(1, 1, 2, 3) = (3)(0, 0, 1, 1) + (-1)(-1, -1, 1, 0)$, so $(1, 1, 2, 3)$ is in the span of $\{(0, 0, 1, 1), (-1, -1, 1, 0)\}$.

Therefore $\{(0, 0, 1, 1), (-1, -1, 1, 0)\}$ is a basis for the span of these four vectors.

(c.) Let v be any vector in \mathbb{R}^4 . Show that v is in W if and only if $v \cdot z = 0$ for every element of Z (where $v \cdot z$ denotes the familiar dot product).

$$\text{Let } S = \{(1, 1, 2, 3), (2, 2, -4, -2), (0, 0, 1, 1), (-1, -1, 1, 0)\}.$$

First, by the definition of W , we see that $v = (w, x, y, z)$ is in W if and only if v satisfies the equations of the system in part (a), which can be rewritten as

$$\begin{aligned}(1, 1, 2, 3) \cdot (w, x, y, z) &= 0 \\ (2, 2, -4, -2) \cdot (w, x, y, z) &= 0 \\ (0, 0, 1, 1) \cdot (w, x, y, z) &= 0 \\ (-1, -1, 1, 0) \cdot (w, x, y, z) &= 0.\end{aligned}$$

So v is in W if and only if $s \cdot v = 0$ for every $s \in S$. Since the dot product is commutative, we can write this as $v \cdot s = 0$ for every $s \in S$.

Now, by definition, $Z = \text{span}(S)$. So we must show that the following two things are equivalent:

- (a.) For every $s \in S$, we have $v \cdot s = 0$.
- (b.) For every $z \in \text{span}(S)$, we have $v \cdot z = 0$.

We will do this for any set $S \subset \mathbb{R}^n$.

First, it is clear that (b) \implies (a), since if $s \in S$ then $s \in \text{span}(S)$, so by (b) we have $v \cdot s = 0$.

To show that (a) \implies (b), assume that (a) holds, and let $z \in \text{span}(S)$. We must show $v \cdot z = 0$. By the definition of span, we can write $z = r_1 s_1 + r_2 s_2 + \cdots + r_n s_n$ for some scalars r_1, r_2, \dots, r_n and elements s_1, s_2, \dots, s_n of S . Now by properties of the dot product, we have

$$v \cdot z = v \cdot (r_1 s_1 + r_2 s_2 + \cdots + r_n s_n) = r_1(v \cdot s_1) + r_2(v \cdot s_2) + \cdots + r_n(v \cdot s_n).$$

By (a) we have $v \cdot s_i = 0$ for $i = 1, 2, \dots, n$, and so we can conclude $v \cdot z = 0$.

(d.) Let A be the matrix $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & -4 & -2 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$, and A^t be the transpose of A . Then L_A

and L_{A^t} are linear transformations from \mathbb{R}^4 to \mathbb{R}^4 . Two pairs of the spaces W , Z , $N(L_A)$, $R(L_A)$, $N(L_{A^t})$, and $R(L_{A^t})$ are equal. Which ones?

$$\begin{aligned}N(L_A) &= W \\ R(L_{A^t}) &= Z.\end{aligned}$$

To see the first equality, note that $N(L_A)$ consists of all solutions to the matrix equation $Ax = 0$, which is equivalent to the system of equations in part (a), and W also consists of all solutions to that system.

To see the second, note that $R(L_{A^t})$ is the span of the columns of A^t , which are the rows of A , and Z is also the span of the rows of A .

The interesting fact that comes out of this (which holds for any matrix A in $M_{m \times n}(\mathbb{R})$, as you can see by looking at the proof in part(c)) is that $R(L_{A^t})$ consists of all vectors that are perpendicular to every vector in $N(L_A)$.

(4.) Suppose that T and U are linear transformations from \mathbb{R}^5 to \mathbb{R}^5 , and $\text{rank}(T) = \text{rank}(U) = 3$.

(a.) What is the largest possible rank of the composition TU ?

The largest possible rank is 3.

Show this rank is possible by giving an example.

Let $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, 0, 0)$, and $U(x_1, x_2, x_3, x_4, x_5) = (x_3, x_2, x_1, 0, 0)$. Both U and T have rank 3 since their range is the three-dimensional subspace spanned by $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, and $(0, 0, 1, 0, 0)$.

Now $TU(x_1, x_2, x_3, x_4, x_5) = T(U(x_1, x_2, x_3, x_4, x_5)) = T(x_3, x_2, x_1, 0, 0) = (x_3, x_2, x_1, 0, 0)$. We see TU has rank 3 since its range is also the three-dimensional subspace spanned by $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, and $(0, 0, 1, 0, 0)$.

Prove that no larger rank is possible.

The range of TU is contained in the range of T . (To see this: Suppose that $v \in R(TU)$. This means that, for some w , we have $v = TU(w) = T(U(w))$, which shows that $v \in R(T)$.) Therefore the dimension of $R(TU)$ is at most the dimension of $R(T)$. But now we have that

$$\text{rank}(TU) = \dim(R(TU)) \leq \dim(R(T)) = \text{rank}(T) = 3.$$

(b.) What is the smallest possible rank of the composition TU ?

The smallest possible rank is 1.

Show this rank is possible by giving an example.

Let $T(x_1, x_2, x_3, x_4, x_5) = (x_3, x_4, x_5, 0, 0)$, and $U(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, 0, 0)$. Both U and T have rank 3 since their range is the three-dimensional subspace spanned by $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, and $(0, 0, 1, 0, 0)$.

Now $TU(x_1, x_2, x_3, x_4, x_5) = T(U(x_1, x_2, x_3, x_4, x_5)) = T(x_1, x_2, x_3, 0, 0) = (x_3, 0, 0, 0, 0)$. We see TU has rank 1 since its range is the one-dimensional subspace spanned by $(1, 0, 0, 0, 0)$.

Prove that no smaller rank is possible.

We need to show that $\text{rank}(TU) \neq 0$. We will suppose that $\text{rank}(TU) = 0$ and get a contradiction.

Because $\text{rank}(TU) = 0$, we have $R(TU) = \{0\}$, so for every vector $v \in V$, we have $TU(v) = 0$, or $T(U(v)) = 0$.

This means that for every vector $w = U(v)$ in $R(U)$, we have $T(w) = T(U(v)) = 0$. This shows that $R(U) \subseteq N(T)$. But we know that $\dim(R(U)) = \text{rank}(U) = 3$, and by the Dimension Theorem,

$$\dim(N(T)) = \text{nullity}(T) = \dim(\text{domain}(T)) - \text{rank}(T) = 5 - 3 = 2.$$

Now we have a three-dimensional subspace ($R(U)$) contained in a two-dimensional subspace ($N(T)$).

This is impossible, so we have a contradiction.

(5.) Suppose that W and Z are subspaces of a vector space V , that $W \cap Z = \{0\}$, and that $\text{span}(W \cup Z) = V$.

Show that if X is a basis for W and Y is a basis for Z , then $X \cup Y$ is a basis for V .

To show $X \cup Y$ is a basis, we need to show that it is linearly independent and spans V .

First we show that $X \cup Y$ is linearly independent. To do this, we need to show that no nontrivial linear combination of vectors from $X \cup Y$ equals zero.

Suppose then that

$$(1) \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n + b_1y_1 + b_2y_2 + \cdots + b_my_m = 0,$$

where $x_i \in X$ and $y_j \in Y$. We must show that $a_i = 0$ and $b_j = 0$ for all i and j . We can rewrite equation (1) as

$$(2) \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = -(b_1y_1 + b_2y_2 + \cdots + b_my_m).$$

Because the left hand side of equation (2) is a linear combination of vectors from X , which is a basis for W , it must be in W ; that is,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \in W.$$

Similarly, the right hand side of equation (2) is a linear combination of vectors from Y , which is a basis for Z , so

$$-(b_1y_1 + b_2y_2 + \cdots + b_my_m) \in Z.$$

We are given that $W \cap Z = \{0\}$, so 0 is the only vector that is in both W and Z . This means we have

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = -(b_1y_1 + b_2y_2 + \cdots + b_my_m) = 0.$$

But now we have

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0,$$

and since the x_i come from the linearly independent set X , we have $a_i = 0$ for all i . Similarly, since

$$b_1y_1 + b_2y_2 + \cdots + b_my_m = 0$$

and the y_j come from the linearly independent set Y , we have $b_j = 0$ for all j .

Now we show that $X \cup Y$ spans V . To do this, we must show that any vector in V can be expressed as a linear combination of vectors from $X \cup Y$.

Let v be any vector of V . Because v is in the span of $W \cup Z$, we can write

$$(3) \quad v = a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1z_1 + b_2z_2 + \cdots + b_mz_m = 0,$$

where $w_i \in W$ and $z_j \in Z$.

Because W is a subspace,

$$a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W,$$

and since any element of W can be expressed as a linear combination of elements of the basis X , we can write

$$a_1w_1 + a_2w_2 + \cdots + a_nw_n = c_1x_1 + c_2x_2 + \cdots + c_kx_k,$$

where $x_i \in X$. Similarly, we can write

$$b_1z_1 + b_2z_2 + \cdots + b_mz_m = d_1y_1 + d_2y_2 + \cdots + d_\ell y_\ell,$$

where $y_j \in Y$. Substituting back into equation (3), we get

$$v = c_1x_1 + c_2x_2 + \cdots + c_kx_k + d_1y_1 + d_2y_2 + \cdots + d_\ell y_\ell,$$

which expresses v as a linear combination of elements from $X \cup Y$.

This completes the proof.

Outline of an alternative proof for (2)(a):

Instead of using the proof given above, you can prove more directly that a square matrix A is invertible if and only if its columns are independent.

First, show that an $n \times n$ matrix A is invertible if and only if there is a matrix B such that $AB = I_n$. To do this, you can use the ideas in exercises 9 and 10a on page 107 of the textbook. A way to do 9 is to relate the properties of A , B , and AB to properties of L_A , L_B , and L_{AB} .

Second, show that the i^{th} column of AB is a linear combination of the columns of A , with the coefficients given by the i^{th} column of B . Therefore, there is a B such that $AB = I_n$ if and only if the columns of I_n (the standard basis vectors) are all in the span of the columns of A .

Now the standard basis vectors are all in the span of the columns of A if and only if the columns of A span F^n , which, since there are n columns and F^n has dimension n , is the case if and only if the columns of A are linearly independent.

In problem (4) we can prove a more general fact: Let V , W , and Z be vector spaces with dimensions m , n , and p respectively. If $U : V \rightarrow W$ and $T : W \rightarrow Z$ are linear transformations with $\text{rank}(U) = r$ and $\text{rank}(T) = s$, then we have

$$\begin{aligned}\text{rank}(TU) &\leq r; \\ \text{rank}(TU) &\leq s; \\ \text{rank}(TU) &\geq 0; \\ \text{rank}(TU) &\geq (r + s) - n.\end{aligned}$$

(Note, by the Dimension Theorem, we know that $r \leq m$, $r \leq n$, $s \leq n$, and $s \leq p$.)

To prove this, you can let \bar{T} be the restriction of T with domain the subspace $R(U) \subseteq W$ and codomain the subspace $R(T) \subseteq Z$. That is, the domain of \bar{T} is $R(U)$, and for any $w \in R(U)$, we have $\bar{T}(w) = T(w)$.

Now, you can prove that the range of TU is the range of \bar{T} . Therefore the problem becomes to find upper and lower bounds on the size of \bar{T} .

By the Dimension Theorem,

$$\begin{aligned}\text{rank}(\bar{T}) &\leq \dim(\text{domain}(\bar{T})) = \dim(R(U)) = \text{rank}(U) = r; \\ \text{rank}(\bar{T}) &\leq \dim(\text{codomain}(\bar{T})) = \dim(R(T)) = \text{rank}(T) = s.\end{aligned}$$

You can also prove that the null space of \bar{T} is $N(T) \cap R(U)$. In particular, $N(\bar{T}) \subseteq N(T)$, so that $\text{nullity}(\bar{T}) \leq \text{nullity}(T)$ and, again by the Dimension Theorem, we have

$$\text{rank}(\bar{T}) = \dim(\text{domain}(\bar{T})) - \text{nullity}(\bar{T}) \geq \dim(\text{domain}(\bar{T})) - \text{nullity}(T) = \dim(R(U)) - (n - \text{rank}(T)) = (r) - (n - s) = (r + s) - n.$$

You can also give examples where the rank is as small and as large as these constraints permit. If we let $\{v_1, \dots, v_m\}$ be a basis for V , $\{w_1, \dots, w_n\}$ be a basis for W , and $\{z_1, \dots, z_p\}$ be a basis for Z , and let $T(v_i) = w_i$ for $i \leq r$ and $T(v_i) = 0$ for $i > r$, we can obtain the largest and smallest possible rank for TU by setting, respectively,

$$U(w_i) = z_i \text{ for } i \leq s \text{ and } U(w_i) = 0 \text{ for } i > s$$

or

$$U(w_i) = 0 \text{ for } i \leq n - s \text{ and } U(w_i) = z_{i-s} \text{ for } i > n - s.$$