Math 24
Winter 2010
Wednesday, January 6

Definition 0.1. A vector space over the real numbers consists of a set V with two operations, addition and scalar multiplication, such that for each two elements x and y in V, there is a unique element $x + y$ in V, for each real number a and each element x in V, there is a unique element ax in V, and the following conditions hold:

(VS 1) For all x and y in V, $x + y = y + x$. (Addition is commutative.)

(VS 2) For all x, y and z in V, $x + (y + z) = (y + x) + z$. (Addition is associative.)

(VS 3) There is an element 0 in V such that for all x in V, $x + 0 = x$. (The element 0 is an additive identity.)

(VS 4) For every element x in V, there is an element $-x$ in V such that $x + (-x) = 0$. (The element $-x$ is an additive inverse for x.)

(VS 5) For all x in V, $1x = x$.

(VS 6) For all real numbers a and b and all x in V, $a(bx) = (ab)x$.

(VS 7) For all real numbers a and all x and y in V, $a(x + y) = ax + ay$. (Scalar multiplication is left-distributive over vector addition.)

(VS 8) For all real numbers a and b and all x in V, $(a + b)x = ax + bx$. (Scalar multiplication is right-distributive over addition of real numbers.)

The elements of V are called vectors, and real numbers are called scalars.

In multivariable calculus, you worked with the vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, and in general, $\mathbb{R}^n$. There are many other kinds of vector spaces. For example, the collection of continuous functions from the closed unit interval $[0, 1]$ to the real numbers, with the usual addition and multiplication by scalars, is a vector space. This space is denoted $\mathcal{C}([0, 1], \mathbb{R})$.

However, the collection of continuous functions from $[0, 1]$ to $[0, 1]$ is not a vector space, because if you add together functions f and g whose range is contained in $[0, 1]$, the range of the sum $f + g$ will not always be contained in $[0, 1]$. In other words, this collection fails to be a vector space because it is not closed under addition.

Exercise 0.2. Find as many different examples of vector spaces over the real numbers as you can. The more different from each other, the better. Here's a hint to get started with: Many of the collections we thought of on Monday that are not fields are in fact vector spaces.

Exercise 0.3. Is the collection of functions $f : \mathbb{R} \to \mathbb{R}$ that are solutions of the differential equation

$$f'' + f = 0$$

a vector space (with the usual addition and scalar multiplication)?

How about the collection of functions $f : \mathbb{R} \to \mathbb{R}$ that are solutions of the differential equation

$$f'' + f = 1?$$

Exercise 0.4. Is the collection of angles α with $0 \leq \alpha < 2\pi$, with addition and scalar multiplication "mod 2π", a vector space? (To compute $\alpha + \beta$ or $r\alpha$, first add or multiply as real numbers, then take the number in the interval $[0, 2\pi)$ that gives the same angle. So for example $\pi + \pi = 0$ and $3(\pi) = \pi$.)

Exercise 0.5. Show that, for any vector space V, the law of cancellation for addition holds: For all x, y, and z in V,

$$x + z = y + z \implies x = y.$$

Exercise 0.6. Show that, for any vector space V, the additive identity is unique. (That is, if 0 and $\overline{0}$ are additive identity elements as in VS 3, then $0 = \overline{0}$.)