Math 24

Winter 2010

Wednesday, February 24

(1.) TRUE or FALSE?

(a.) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

(b.) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

(c.) There exists a square matrix with no eigenvectors.

(d.) Eigenvalues must be nonzero scalars.

(e.) Any two eigenvectors are linearly independent.

(f.) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(g.) Linear operators on infinite-dimensional vector spaces never have eigenvalues.

(h.) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.

(i.) Similar matrices always have the same eigenvalues.

(j.) Similar matrices always have the same eigenvectors.

(k.) The sum of two eigenvectors of an operator T is always an eigenvector of T.

(l.) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(m.) Two distinct eigenvalues corresponding to the same eigenvalue are always linearly dependent.

(n.) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvalue of T.

(o.) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

(p.) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose j^{th} column is v_n $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(q.) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(r.) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

(s.) You can always tell from the characteristic polynomial of A whether A is diagonalizable.

(t.) You can sometimes tell from the characteristic polynomial of A whether A is diagonalizable.

(u.) You can always tell from the characteristic polynomial of A whether A is invertible.

(2.) Find an invertible matrix Q and find a diagonalizable matrix B such that either $QAQ^{-1} = B$ or $Q^{-1}AQ = B$. Be sure to say which of these two equations holds for your Q and B.

$$A = \begin{pmatrix} 1 & -7 & 2 \\ 0 & 2 & 0 \\ 0 & -10 & 2 \end{pmatrix}.$$

(3.) For the matrix A in problem (2), find a basis for the eigenspace of A corresponding to each eigenvalue. Describe each of these eigenspaces geometrically. (Be specific. Don't just say "a line"; specify which line.)

(4.) Test the matrix A for diagonalizability.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(5.) Suppose a linear operator T on an n-dimensional vector space V has only one eigenvalue $\lambda = 1$, and T is diagonalizable. What can you conclude about T?

What can you say in general about diagonalizable linear operators with a single eigenvalue?

(6.) Show that if T is a diagonalizable linear operator on an n-dimensional vector space V with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then each vector v in V can be expressed uniquely as

$$v = v_1 + v_2 + \cdots + v_k$$

where $v_i \in E_{\lambda_i}$.