Math 24
Winter 2010
Monday, February 22

(3.) Find the eigenvalues, and a basis for each of the corresponding eigenspaces, of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of this matrix?

The characteristic polynomial of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ is $det(A - tI) = det \begin{pmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 1 \\ 0 & 0 & -1-t \end{pmatrix} = (1-t)^2(-1-t)$. The eigenvalues of A are the roots of this polynomial, 1 (multiplicity 2) and -1 (multiplicity 1).

To find the eigenspace corresponding to $\lambda = -1$, we find the null space of $A - \lambda I$, or $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Solving $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, so a basis for the eigenspace is $\left\{ \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$.

To find the eigenspace corresponding to $\lambda = 1$, we find the null space of $A - \lambda I$, or $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}$. Solving $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so a basis for the eigenspace is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Since the eigenvalue 1 has multiplcity 2, but its eigenspace has dimension 1, A is not diagonalizable and there is no basis of eigenvectors.