Math 24
Winter 2010
Wednesday, February 17

(1.) TRUE or FALSE?

(a.) If E is an elementary matrix, then $det(E) = \pm 1$.

FALSE. For example, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is an elementary matrix, obtained from the identity matrix by the elementary row operation of multiplying row 1 by 2.

(b.) For any $A, B \in M_{n\times n}(F)$, $det(AB) = (det(A))(det(B))$.

TRUE.

(c.) A matrix $A \in M_{n\times n}(F)$ is invertible if and only if $det(A) = 0$.

FALSE. A is invertible if and only if $det(A) \neq 0$.

(d.) A matrix $A \in M_{n\times n}(F)$ has rank n if and only if $det(A) \neq 0$.

TRUE.

(e.) For any $A \in M_{n\times n}(F)$, $det(A^t) = -det(A)$.

FALSE. $det(A^t) = det(A)$.

(f.) The determinant of a square matrix can be evaluated by cofactor expansion along any column.

TRUE.

(g.) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

FALSE. Cramer's rule cannot be used unless the coefficient matrix has nonzero determinant.

(h.) Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{det(M_k)}{det(A)} \text{ for } k = 1, 2, \ldots, n.$$

FALSE. This question is about using Cramer's rule with rows instead of columns. In order for this to work, you need to use not A but A^t, along with b^t.

(i.) If Q is an invertible matrix, then $det(Q^{-1}) = \dfrac{1}{det(Q)}$.

TRUE. This follows from the fact that $det(Q) \cdot det(Q^{-1}) = det(QQ^{-1}) = det(I) = 1$.

(j.) The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries. (A matrix is lower triangular if the only nonzero entries are on or below the main diagonal.)

TRUE. This follows from the fact that the transpose of a lower triangular matrix is an upper triangular matrix with the same diagonal entries.

(2.) Let A be an $n \times n$ matrix, and k a scalar. Find the determinant of kA in terms of the determinant of A.

To transform A to kA using elementary row operations, you multiply each row of A by k. Each time you multiply a row by k, you also multiply the determinant by k. Therefore

$$det(kA) = k^n \, det(A).$$

(3.) Show that if A and B are similar $n \times n$ matrices, then $det(A) = det(B)$.

The definition of similar states that A and B are similar if and only if we can express A as $A = Q^{-1}BQ$ for some invertible matrix Q.
In that case, we can express the determinant of A as

$$det(A) = det(Q^{-1}BQ) = det(Q^{-1})det(B)det(Q) = \frac{1}{det(Q)}det(B)det(Q) = det(B).$$

(4.) Suppose that $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},$$

where A is a square matrix, 0 is a zero matrix, and I is an $m \times m$ identity matrix. Prove that $det(M) = det(A)$.

There are two ways to do this. One is to use induction on m, for the inductive step, computing the determinant by expanding along the last row. Another is to note that you can use row operations to convert A to upper triangular form, and to do this you need only operate with the first $n - m$ rows of M.

(5.) Let $A \in M_{n\times n}(F)$ be nonzero. For any m with $1 \leq m \leq n$, an $m \times m$ submatrix is obtained by deleting $n - m$ rows and $n - m$ columns of A. For example, if we start with $A = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 8 \\ -2 & 0 & 0 & -4 \\ 1 & 4 & 4 & 10 \end{pmatrix}$ and delete rows 2 and 3 and columns 2 and 4, we get the 2×2 submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$.

(a.) Show that if A is an $n \times n$ matrix and there is a $k \times k$ submatrix of A with nonzero determinant, then $rank(A) \geq k$.

An idea for one possible proof is this: If there is a $k \times k$ submatrix with nonzero determinant, then the columns of that submatrix are linearly independent. You can show that the columns of A used in that submatrix are also linearly independent. The number of these columns is k, so $rank(A) \geq k$.

(b.) Show that if A is an $n \times n$ matrix with rank k, then there is a $k \times k$ submatrix of A with nonzero determinant.

Here you can use the same idea backwards: There are k linearly independent columns of A. Eliminate the other columns of A to get an $n \times k$ matrix B in which all the columns are linearly independent. Since B has rank k, we know B has k linearly independent rows. Eliminate the other rows to get a $k \times k$ submatrix of A with rank k, hence with nonzero determinant.