Math 24

Winter 2010

Wednesday, February 10

(1.) TRUE or FALSE?

(a.) Any system of linear equations has at least one solution.

FALSE.

(b.) Any system of linear equations has at most one solution.

FALSE.

(c.) Any homogeneous system of linear equations has at least one solution.

TRUE. We always have 0 as a solution.

(d.) Any system of $n$ linear equations in $n$ unknowns has at most one solution.

FALSE.

(e.) Any system of $n$ linear equations in $n$ unknowns has at least one solution.

FALSE.

(f.) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

FALSE.

(g.) If the coefficient matrix of a homogeneous system of $n$ linear equations in $n$ unknowns is invertible, then the system has no nonzero solutions.

TRUE. Any system of $n$ linear equations in $n$ unknowns whose coefficient matrix is invertible has exactly one solution.

(h.) The solution set of any system of $m$ linear equations in $n$ unknowns is a subspace of $F^n$.

FALSE. If the system is homogeneous, the solution set is a subspace. Otherwise, it is a coset of the solution space of the corresponding homogeneous system.

(2.) Let $A$ denote the coefficient matrix of the following system of linear equations. Compute $A^{-1}$, and use $A^{-1}$ to solve the system.

$$x + 2y - z = 5$$
$$x + y + z = 1$$
$$2x - 2y + z = 4$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix}.$$

We can rewrite our system as
$$A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\1\\4\end{pmatrix},$$
and multiply both sides of the equation on the left by $A^{-1}$ to get
$$\begin{pmatrix}x\\y\\z\end{pmatrix}=A^{-1}\begin{pmatrix}5\\1\\4\end{pmatrix},$$
Using row reduction, we can compute $A^{-1}=\begin{pmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9}\\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9}\end{pmatrix}$, and so
$$\begin{pmatrix}x\\y\\z\end{pmatrix}=A^{-1}\begin{pmatrix}5\\1\\4\end{pmatrix}=\begin{pmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9}\\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9}\end{pmatrix}\begin{pmatrix}5\\1\\4\end{pmatrix}=\begin{pmatrix}3\\0\\-2\end{pmatrix}.$$
We can check this answer by going back to the original system of linear equations.

(3.) Find the set of solutions for the homogeneous system associated to the following system of linear equations, and determine whether this system has a solution.

$$x+2y-z=1$$
$$2x+y+2z=3$$
$$x-4y+7z=4$$

We rewrite this system as
$$\begin{pmatrix}1 & 2 & -1\\ 2 & 1 & 2\\ 1 & -4 & 7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\4\end{pmatrix},$$
and row reduce the augmented matrix of the system,
$$\begin{pmatrix}1 & 2 & -1 & | & 1\\ 2 & 1 & 2 & | & 3\\ 1 & -4 & 7 & | & 4\end{pmatrix},$$
to get
$$\begin{pmatrix}1 & 0 & \frac{5}{3} & | & \frac{5}{3}\\ 0 & 1 & -\frac{4}{3} & | & -\frac{1}{3}\\ 0 & 0 & 0 & | & 1\end{pmatrix}.$$

Since the augmented matrix has higher rank than the coefficient matrix, there are no solutions.

We can see that the augmented matrix of the homogenous system

$$x + 2y - z = 0$$
$$2x + y + 2z = 0$$
$$x - 4y + 7z = 0$$

would row reduce to the form

$$\begin{pmatrix} 1 & 0 & \frac{5}{3} & | & 0 \\ 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix},$$

and so the homogeneous system is equivalent to the system

$$x + \tfrac{5}{3}z = 0$$
$$y - \tfrac{4}{3}z = 0,$$

or to the system

$$x = -\tfrac{5}{3}z$$
$$y = \tfrac{4}{3}z,$$

which has the solution set

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} -\frac{5}{3} \\ \\ \frac{4}{3} \\ \\ 1 \end{pmatrix},$$

where $r$ can be any scalar.

(4.) Find the set of solutions for the following system of linear equations. Note that the coefficient matrix is the same as in problem (3), and one obvious solution is $x = 1$, $y = z = 0$.

$$x + 2y - z = 1$$
$$2x + y + 2z = 2$$
$$x - 4y + 7z = 1$$

We know the solution set of this system is arrived at by adding any one particular solution to the solution set of the associated homogeneous system. We are given a particular solution, and we found a solution to the homogeneous system in the last problem, so our solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} -\frac{5}{3} \\ \\ \frac{4}{3} \\ \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where $r$ can be any scalar.

(5.) Prove or give a counterexample to the following statement: If the coefficient matrix of a system of $m$ linear equations in $n$ unknowns has rank $m$, then the system has a solution.

We can prove the statement.

If the coefficient matrix of a system of $m$ linear equations in $n$ unknowns is the matrix $A$, then $A$ is $m \times n$, so $L_A : F^n \to F^m$, and the fact that the rank of $A$ is $m$ tells us that $L_A$ is onto.

To find a solution to a system of linear equations with coefficient matrix $A$ is the same as finding a solution to a matrix equation $Ax = b$, which we can rewrite as $L_A(x) = b$. Because $L_A$ is onto, $b$ is in its range, so there must be some $s$ such that $L_A(s) = b$, or $As = b$. Then $s$ is a solution to the system.

(6.) Prove or give a counterexample to the following statement: If $m < n$, then a system of $m$ linear equations in $n$ unknowns (over the field $\mathbb{R}$) has either no solutions or infinitely many solutions.

We can prove the statement.

If the coefficient matrix of a system of $m$ linear equations in $n$ unknowns is the matrix $A$, then $A$ is $m \times n$, so $L_A : \mathbb{R}^n \to \mathbb{R}^m$. From the fact that $m < n$ and the Dimension Theorem, we can see that $nullity(L_A) > 0$, so that $N(L_A)$ is a subspace of $\mathbb{R}^n$ having dimension at least 1. Therefore there are infinitely many elements in $N(L_A)$, which is the solution set of the homogeneous system with coefficient matrix $A$.

Now if a system with coefficient matrix $A$ has some solution $s$, then the solution set of the system is the coset $s + N(L_A)$. Since $N(L_A)$ has infinitely many elements, so do its cosets.