Math 24
Winter 2010
Friday, February 5

(1.) TRUE or FALSE?

(a.) An elementary matrix is always square.

TRUE. An elementary matrix is obtained from the $n \times n$ identity matrix by an elementary row or column operation.

(b.) The only entries of an elementary matrix are zeros and ones.

FALSE. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is elementary (obtained from I_2 by multiplying row 1 by 2).

(c.) The $n \times n$ identity matrix is an elementary matrix.

TRUE. It can be obtained from I_n by multiplying a row or column by 1.

(d.) The product of two $n \times n$ matrices is an elementary matrix.

FALSE. Even if the two matrices are themselves elementary. Consider $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. There is no way to obtain this from I_2 by a single elementary operation.

(e.) The inverse of an elementary matrix is an elementary matrix.

TRUE. It corresponds to the inverse elementary operation.

(f.) The transpose of an elementary matrix is an elementary matrix.

TRUE. If the original matrix corresponds to an elementary operation on rows, the transpose corresponds to the same operation on columns.

(g.) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

FALSE. The matrix $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ can be obtained from $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ by a single row operation, but not by any number of column operations. Any column operation will still leave columns that are multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(h.) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

TRUE. Use the inverse elementary row operation.

(2.) We showed in class that if $AX = B$ is a matrix equation, where X is a variable (an "unknown"), and A and B are converted to A^ρ and B^ρ by using the same elementary row operation, then the matrix equation $A^\rho X = B^\rho$ has the same solutions as the original equation $AX = B$.

(a.) Use this fact to find $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ by solving the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Try to convert it into an equation of the form $IX = B$, where I is the identity matrix.)

First add (-1) times row 2 to row 1:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then add (-1) times row 1 to row 2:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Thus the solution is $X = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, and this is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$.

(b.) Write both $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ as products of elementary matrices. (Recall that performing row operation ρ on a matrix A is the same as multiplying A on the left by the corresponding elementary matrix $E = I^\rho$.)

We look at the steps we performed on the identity matrix above, using this fact.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

First add (-1) times row 2 to row 1:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then add (-1) times row 1 to row 2:

$$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This tells us that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now we can use the fact that $(AB)^{-1} = B^{-1}A^{-1}$:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \right)^{-1} = \left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}.$$

The inverse of an elementary matrix corresponding to some row operation (for example, adding (-1) times row 2 to row 1) is the elementary matrix corresponding to the inverse row operation (in the same example, adding (1) times row 2 to row 1). Therefore we can continue.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(3.) Show that $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & 3 & 3 \end{pmatrix}$ is not invertible, by using elementary row operations to convert the matrix equation

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & 3 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

into an equivalent equation that you can show has no solutions. (Two equations are equivalent if they have exactly the same solutions.)

In the left hand matrix, row 1 plus row 2 equals row 3. Therefore, adding (-1) times row 1 to row 3, then adding (-1) times row 2 to row 3, converts this equation to

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

We can see this equation has no solutions, because if row 3 of matrix A consists entirely of zeros, then row 3 of any product AB must also consist entirely of zeroes (because the entries are the dot products of row 3 of A with the columns of B).

In general, if you attempt to find the inverse of a matrix using the method of problem (2), and the matrix has no inverse, you will end up converting the equation to an equivalent equation that clearly has no solutions.

(4.) Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 (interchange two rows) and 3 (add a multiple of one row to another row) that transform A into an upper triangular matrix. (A matrix B is upper triangular if all entries below the major diagonal, that is, all entries B_{ij} for $i > j$, are zero.)

The idea here is to interchange rows to guarantee that the first entry in the first row is not zero (unless the first column consists entirely of zeroes, in which case we go on to the second entry), then subtract multiples of the first row from all the other rows so they have zero as their first entry. Now we interchange rows to guarantee that the second entry in the second row is not zero (unless all the entries in the second column except the entry in row 1 are zero), then subtract multiples of the second row from later rows to ensure they all have zeroes in the second column. We continue until we are out of rows (or columns).

Formally, we should structure this as a proof by induction: Prove by induction on $k \leq min(m,n)$ (where $min(m,n)$ denotes whichever of m and n is smaller) that we can transform A into a matrix B where for all $j \leq k$, all entries B_{ij} for $i > j$ are zero.

(5.) Let ρ be an elementary row operation and let A^ρ denote the matrix obtained by performing that row operation on A. Show that the function $T : M_{n\times m}(F) \to M_{n\times m}(F)$ defined by $T(A) = A^\rho$ is a linear function.

We know there is an elementary matrix E such that $A^\rho = EA$, so we have $T(A) = EA$. Now we can easily show that T is linear:

$T(A+B) = E(A+B) = EA + EB = T(A) + T(B)$, and
$T(rA) = E(rA) = r(EA) = rT(A)$.

(6.) Prove that any $n \times n$ elementary matrix can be obtained from the identity matrix I_n in at least two ways, either by an elementary row operation or by an elementary column operation.