Math 24
Winter 2010
Friday, February 5

(1.) TRUE or FALSE?

(a.) An elementary matrix is always square.

(b.) The only entries of an elementary matrix are zeros and ones.

(c.) The $n \times n$ identity matrix is an elementary matrix.

(d.) The product of two $n \times n$ matrices is an elementary matrix.

(e.) The inverse of an elementary matrix is an elementary matrix.

(f.) The transpose of an elementary matrix is an elementary matrix.

(g.) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

(h.) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

(2.) We showed in class that if $AX = B$ is a matrix equation, where X is a variable (an "unknown"), and A and B are converted to A^ρ and B^ρ by using the same elementary row operation, then the matrix equation $A^\rho X = B^\rho$ has the same solutions as the original equation $AX = B$.

(a.) Use this fact to find $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ by solving the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Try to convert it into an equation of the form $IX = B$, where I is the identity matrix.)

(b.) Write both $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ as products of elementary matrices. (Recall that performing row operation ρ on a matrix A is the same as multiplying A on the left by the corresponding elementary matrix $E = I^\rho$.)

(3.) Show that $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & 3 & 3 \end{pmatrix}$ is not invertible, by using elementary row operations to convert the matrix equation

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & 3 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

into an equivalent equation that you can show has no solutions. (Two equations are equivalent if they have exactly the same solutions.)

(4.) Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 (interchange two rows) and 3 (add a multiple of one row to another row) that transform A into an upper triangular matrix. (A matrix B is upper triangular if all entries below the major diagonal, that is, all entries B_{ij} for $i > j$, are zero.)

(5.) Let ρ be an elementary row operation and let A^ρ denote the matrix obtained by performing that row operation on A. Show that the function $T : M_{n\times m}(F) \to M_{n\times m}(F)$ defined by $T(A) = A^\rho$ is a linear function.

(6.) Prove that any $n \times n$ elementary matrix can be obtained from the identity matrix I_n in at least two ways, either by an elementary row operation or by an elementary column operation.