Math 24
Winter 2010
Wednesday, January 27

(1.) TRUE or FALSE? In each part, V, W, and Z denote finite-dimensional vector spaces with ordered bases α, β and γ respectively, $T : V \to W$ and $U : W \to Z$ denote linear transformations, and A and B denote matrices.

(a.) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$.

FALSE, but $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

(b.) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$.

TRUE. This is the heart of what is going on here. We go from applying a linear transformation to multiplying by a matrix, by representing vectors using their coordinates with respect to a chosen basis.

(c.) $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$.

FALSE, but $[U(w)]_\gamma = [U]_\beta^\gamma [w]_\beta$ for all $w \in W$. Here $[U(w)]_\beta$ and $[U]_\alpha^\beta$ don't even make sense; $U(w)$ is a vector in Z, but β is not a basis for Z.

(d.) $[I_V]_\alpha = I$.

TRUE. The matrix of the identity transformation is the identity matrix.

(e.) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$.

FALSE. Here T^2 doesn't make sense; you can't compute $T(T(v))$, because $T(v)$ is in W, not in V. However, if $U = V = W$ and $\alpha = \beta = \gamma$, then $[T^2]_\alpha^\alpha = ([T]_\alpha^\alpha)^2$, or using the shorter notation, $[T^2]_\alpha = ([T]_\alpha)^2$.

(f.) $A^2 = I$ implies that $A = I$ or $A = -I$.

FALSE. There is a counterexample in the text, and you will find another in problem (5).

(g.) $T = L_A$ for some matrix A.

FALSE. However, if $V = F^n$ and $W = F^m$ this is true, and if α and β are the standard bases, then $T = L_{[T]_\alpha^\beta}$.

(h.) $A^2 = 0$ implies that $A = 0$, where 0 denotes the zero matrix.

FALSE. There is a counterexample in the textbook. Once you have done problem (5), you might think about how you would produce a counterexample.

(i.) $L_{A+B} = L_A + L_B$.

TRUE. The function from $M_{m\times n}(F)$ to $\mathcal{L}(F^n, F^m)$ that sends the matrix A to the function L_A is a linear transformation. It is also one-to-one (L_A is the zero function only if A is the zero matrix) and onto (every element of $\mathcal{L}(F^n, F^m)$ is L_A for some matrix A).

(j.) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

TRUE.

(2.) If $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}$, which of the matrix products AB and BA is defined?

Find the second column of that matrix product.

BA is defined. Rows from the left-hand matrix must match up with columns from the right-hand matrix. B is 3×3 and A is 3×2 so the product BA will be 3×2.

The second column of BA is B times the second column of A, which is

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3(5) + (-2)(1) + 0(2) \\ 1(5) + (-1)(1) + 4(2) \\ 5(5) + 5(1) + 3(2) \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \\ 36 \end{pmatrix}.$$

Notice that the entry in row i is the dot product of the i^{th} row of B with the column vector $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$, which is the second column of A.

(3.) Write down a matrix A such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 2y + z \\ x - 2z \end{pmatrix}$.

$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$. Again, the entries of the product should be the dot product of the corresponding rows of A with the column vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This time, we are multiplying a 2×3 matrix by a 3×1 matrix to obtain a 2×1 matrix.

(4.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates counterclockwise around the origin by ninety degrees (so if v is on the positive x axis, then $T(v)$ is on the positive y-axis), and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects every point perpendicularly onto the x-axis. Let β be the standard basis for $\mathbb{R}^2$.

(a.) Find explicit expressions for $T(x, y)$ and $U(x, y)$. Use these to write explicit expressions for $UT(x, y)$ and $TU(x, y)$. (Recall that $UT(x, y)$ denotes $U(T(x, y))$.) Find $UT(1, 0)$, $UT(0, 1)$, $TU(1, 0)$ and $TU(0, 1)$. Use these values to write down the matrices $[UT]_\beta$ and $[TU]_\beta$.

$T(x, y) = (-y, x)$. $U(x, y) = (x, 0)$.
$UT(x, y) = U(T(x, y)) = U(-y, x) = (-y, 0)$.
$TU(x, y) = T(U(x, y)) = T(x, 0) = (0, x)$.
$UT(1, 0) = (0, 0)$, $UT(0, 1) = (-1, 0)$, $TU(1, 0) = (0, 1)$, $TU(0, 1) = (0, 0)$.
Using the fact that the columns of the matrix of a linear transformation are the coordinates of the images of the basis vectors, we see $[UT]_\beta = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $[TU]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(b.) Find $U(1, 0)$, $U(0, 1)$, $T(1, 0)$ and $T(0, 1)$. Use these values to write down the matrices $[U]_\beta$ and $[T]_\beta$.

$U(1, 0) = (1, 0)$, $U(0, 1) = (0, 0)$, $T(1, 0) = (0, 1)$, $T(0, 1) = (-1, 0)$
$[U]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $[T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(c.) Use matrix multiplication to compute $[U]_\beta[T]_\beta$ and $[T]_\beta[U]_\beta$. Compare with your answers to part (a); did you get what you should?

$[U]_\beta[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.
$[T]_\beta[U]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
$[T]_\beta[U]_\beta = [TU]_\beta$ and $[U]_\beta[T]_\beta = [UT]_\beta$, which is what we should get.

(d.) Compute the matrix product $[U]_\beta[T]_\beta \begin{pmatrix} x \\ y \end{pmatrix}$. This should give you $[UT(x, y)]_\beta$. Compare this with your answer to part (a); did you get what you should?

$[U]_\beta[T]_\beta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} -y \\ 0 \end{pmatrix}$.
We computed $UT(x, y) = (-y, 0)$, so setting $v = (x, y)$, we have $[U]_\beta[T]_\beta[v]_\beta = [UT(v)]_\beta$, which is what we should get.

(5.) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ other than $I_{\mathbb{R}^2}$ and $-I_{\mathbb{R}^2}$, with the property that $T(T(v)) = v$ for every v in $\mathbb{R}^2$. Use T to find a matrix A such that $A \neq I$ and $A \neq -I$ but $A^2 = I$. (Recall that $I_{\mathbb{R}^2}$ denotes the identity transformation on $\mathbb{R}^2$, so $I_{\mathbb{R}^2}(v) = v$ and $-I_{\mathbb{R}^2}(v) = -v$. Your function T should be different from either of these.)

Any reflection has this property. For example, the reflection across the line $x = y$ is given by the linear transformation $T(x, y) = (y, x)$. It is easy to see that $T^2(x, y) = T(T(x, y)) = (x, y)$ so T^2 is the identity transformation.

Since $T(1, 0) = (0, 1)$ and $T(0, 1) = (1, 0)$, the matrix of T in the standard basis is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since A is the matrix of T, then A^2 should be the matrix of T^2, which is the identity matrix I. We can easily check this by matrix multiplication: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.