Math 24

Winter 2010

Friday, January 22

(4.) Suppose that V and W are vector spaces with subspaces V_1 and W_1 respectively, and that $T : V \to W$ is a linear transformation. Prove that $T(V_1) = \{T(v) \mid v \in V_1\}$ is a subspace of W, and that $T^{-1}(W_1) = \{v \mid T(v) \in W_1\}$ is a subspace of V.

Notice that if $V_1 = V$ then $T(V_1) = R(T)$, and if $W_1 = \{0\}$ then $T^{-1}(W_1) = N(T)$. So the fact that $N(T)$ and $R(T)$ are subspaces of the domain and range of T is a special case of this result.

(5.) Give an example of an infinite dimensional vector space V and a linear transformation $T : V \to V$ such that T is one-to-one but not onto. Also give such an example where T is onto but not one-to-one.

We know of a couple of infinite dimensional vector spaces over $\mathbb{R}$ that are fairly concrete. One is $P(\mathbb{R})$, the space of all polynomials with real coefficients. Another is the space of infinite sequences of real numbers.

(6.) Suppose that $\beta = \{u_1, \ldots, u_k, w_1, \ldots, w_j\}$ is a basis for the n-dimensional vector space V. Define two subspaces of V by $U = span(\{u_1, \ldots, u_k\})$ and $W = span(\{w_1, \ldots, w_k\})$.

(a.) Show that any vector $v \in V$ can be written uniquely as $v = u + w$ where $u \in U$ and $w \in W$.

(You might want first to show that any vector can be written in this form, and then show that there is only one way to do it. That is, first show existence and then show uniqueness.)

(b.) Define a function $T : V \to V$ as follows: For $v = u + w$ where $u \in U$ and $w \in W$, define $T(v) = u$. (Notice that by part (a), the function T is in fact well-defined.) Show that T is linear, and find $N(T)$ and $R(T)$. Verify the dimension theorem in this case. (That is, show that $rank(T) + nullity(T) = dim(V)$.)

(c.) Given a linear transformation $T : V \to V$ (where V is n-dimensional), exactly when can T be written in this form? Prove your answer is correct.