Basis and Dimension (cont'd)

January 24, 2007

Direct Sum

- If S_1 and S_2 are nonempty subsets of a vector space V, then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } x \in S_2\}$.

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- A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \bigcap W_2 = \{0\}$ and $W_1 + W_2 = V$.

- We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Finite-dimensional Vector Spaces

- A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors.

- The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by $\dim(V)$.

- A vector space that is not finite-dimensional is called **infinite-dimensional**.

Theorem. [Replacement Theorem] *Let Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \bigcup H$ generates V.*

Corollary. *Let V be a vector space with dimension n.*

1. *Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.*

2. *Any linearly independent subset of V that contains exactly n vectors is a basis for V.*

3. *Every linearly independent subset of V can be extended to a basis for V.*

The Dimension of Subspaces

Theorem. *Let W be a subspace of a finite-dimensional vector space V. The W is finite-dimensional and dim$(W) \leq$ dim(V). Moreover, if dim$(W) =$ dim(V), then $V = W$.*

Corollary. *If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.*