Linear Dependence and Linear Independence (cont'd)

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Linear Dependence and Linear Independence

- A subset S of a vector space V is called **linearly dependent** if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0.$$

  In this case we say that the vectors of S are linearly dependent.

- A subset S of V that is not linearly dependent is called **linearly independent**. We say that the vectors of S are linearly independent.

Results about linear dependence and linear independence

Theorem. *Let V be a vector space, and let $S_1 \subset S_2 \subset V$. If S_1 is linearly dependent, then S_2 is linearly dependent.*

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Theorem. *Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S\bigcup\{v\}$ is linearly dependent if and only if $v \in$ span(S).*

Bases and Dimension

- A **basis** β for a vector space V is a linearly independent subset of V that generates V. If β is a bass for V, we also say that the vectors of β form a basis for V.

Theorem. *Let* V *be a vector space and* $\beta = \{u_1, u_2, \ldots, u_n\}$ *be a subset of* V. *Then* β*is a basis for* V *if and only if each* $v \in V$ *can be uniquely expressed as a linear combination of vectors of* β, *that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Theorem. *If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.*

Theorem. [Replacement Theorem] *Let Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \bigcup H$ generates V.*