Gram-Schmidt Orthogonalization Process

Lecture 25

March 7, 2007

Orthogonal Vectors

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5. A subset S of V is an **orthonormal basis** for V if it is an ordered basis that is orthonormal.

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- *If S is an orthogonal subset of V consisting of nonzero vectors then S is linearly independent.*
- *Any finite dimensional inner product space has an orthonormal basis.*
- *If $\beta = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for V, then for any $x \in V$*

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$

The coefficients $\langle x, v_i \rangle$ are called the Fourier coefficients.

Gram-Schmidt Orthogonalization Process

Theorem

Let V be an inner product space and $S = \{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n.$$

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- *proving that the first statement in the infinite sequence of statements is true, and then*
- *proving that if any one statement in the infinite sequence of statements is true, then so is the next one.*

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1. The basis: showing that the statement holds when $n = 0$.
2. The inductive step: showing that if the statement holds for $n = m$, then the same statement also holds for $n = m + 1$.

Mathematical Induction: The First Example

Example

Show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Back to the Gram-Schmidt Process

Theorem

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$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n.$$

Thank you and good luck!

The End!