Elementary Matrix Operations and Elementary Matrices

Lecture 15

February 12, 2007

Definition

Let A be an $m \times n$ matrix. The left multiplication by A is the linear transformation $L_A : F^n \to F^m$ defined by

$$L_A(x) = Ax.$$

Left-Multiplication Transformations

Theorem

Let A and B be $n \times m$ matrices. Then

1. $[L_A]_\beta^\gamma = A$.
2. $L_A = L_B$ *if and only if* $A = B$.
3. $L_{A+B} = L_A + L_B$ *and* $L_{aA} = aL_a$ *for all* $a \in F$.
4. *If* $T : F^n \to F^m$ *is linear, then there exists a unique* $m \times n$ *matrix* C *such that* $T = L_C$.
5. $L_{AE} = L_A L_E$.

Change of Coordinates for Left-Multiplication Transformations

Theorem

Let A be an $n \times n$ matrix and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.

Elementary Matrix Operations

Definition

Let A be an $m \times n$ matrix. Any one of the following three operations on the rows [columns] of A is called an **elementary row [column] operation:**

1. interchanging any two rows [columns] of A. (**type 1**)
2. multiplying any row [column] of A by a nonzero scalar.(**type 2**)
3. adding any scalar multiple of a row [column] of A to another row [column].(**type 3**)

Elementary Matrices

Definition

- An $n \times n$ **elementary matrix** is a matrix obtained by performing an elementary operation on I_n.
- The elementary matrix is said to be of **type 1, 2,** or 3 according to whether the elementary operation performed on I_n is a type $1, 2$, or 3 operation, respectively.

Multiplying with an Elementary Matrix

Theorem

Let $A \in M_{m \times n}(F)$, and suppose that B is obtained from A by performing an elementary row operation. Then there exists an $m \times m$ elementary matrix such that $B = EA$. In fact, E is obtained from I_m by performing the same row operation as that which was performed on A to obtain B.
Conversely, if E is an elementary $m \times m$ matrix, then EA is the matrix obtained from A by performing the same elementary row operation which produces E from I_m.

Every Elementary Matrix is Invertible

Theorem

Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.