The Change of Coordinate Matrix

Lecture 14

February 7, 2007

Linear Maps "are" Matrices

Theorem

Let V and W be finite-dimensional vector spaces over F of dimensions n and m, and let β and γ be ordered bases for V and W, respectively. Then the function $\phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$, defined by

$$\phi(T) = [T]_\beta^\gamma$$

is an isomorphism.

Change of Variables

Example

1. Solve the integral $\int \cos(x)e^{\sin(x)}dx$

Change of Variables

Example

1. Solve the integral $\int \cos(x)e^{\sin(x)}\mathrm{d}x$
2. Can you recognize the shape of the curve given by the equation

$$2x^2 - 4xy + 5y^2 = 1?$$

Theorem

Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then

1. *Q is invertible.*
2. *For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.*

Change of Coordinate Matrix

Definition

The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called a change of coordinate matrix.

Fact

If $\beta = \{x_1, x_2, \ldots, x_n\}$ *and* $\beta' = \{x_1', x_2', \ldots, x_n'\}$, *then*

$$x_j' = \sum_{i=1}^{n} Q_{ij} x_i.$$

Change of the Matrix of a Linear Transformation

Theorem

Let T be a linear operator on a finite-dimensional vector space, and let β and $\beta^{'}$ be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes $\beta^{'}$-coordinates into β-coordinates. Then

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

Similarity

Definition

Let A and B be matrices in $M_{n\times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

Fact

The previous theorem says that $[T]_{\beta'}$ and $[T]_{\beta}$ are similar.

Left-Multiplication Transformations

Definition

Let A be an $m \times n$ matrix. The left multiplication by A is the linear transformation $L_A : F^n \to F^m$ defined by

$$L_A(x) = Ax.$$

Left-Multiplication Transformations

Theorem

Let A and B be $n \times m$ matrices. Then

1. $[L_A]_\beta^\gamma = A$.
2. $L_A = L_B$ *if and only if* $A = B$.
3. $L_{A+B} = L_A + L_B$ *and* $L_{aA} = aL_a$ *for all* $a \in F$.
4. *If* $T : F^n \to F^m$ *is linear, then there exists a unique* $m \times n$ *matrix* C *such that* $T = L_C$.
5. $L_{AE} = L_A L_E$.

Change of Coordinates for Left-Multiplication Transformations

Theorem

Let A be an $n \times n$ matrix and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.