Linear Transformations

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Linear Transformations are Determined by the Values on a Basis

Theorem. *Let* V *and* W *be vector spaces over* F*, and suppose that* $\{v_1, v_2, \ldots, v_n\}$ *is a basis for* V*. For* $w_1, w_2, \ldots, w_n$ *in* W*, there exists exactly one linear transformation* $T : V \to W$ *such that* $T(v_i) = w_i$ *for* $i = 1, 2, \ldots, n$*.*

Corollary. *Let* V *and* W *be vector spaces, and suppose that* V *has a finite basis* $\{v_1, v_2, \ldots, v_n\}$*. If* $U, T : V \to W$ *are linear and* $U(v_i) = T(v_i)$ *for* $i = 1, 2, \ldots, n$*, then* $U = T$*.*

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- Let V be a finite-dimensional vector space.
- An **ordered basis** for V is a basis for V endowed with a specific order.
- We call $\{e_1, e_2, \ldots, e_n\}$ the **standard ordered basis** for F^n.
- We call $\{1, x, x^2, \ldots, x^n\}$ the **standard ordered basis** for $P_n(F)$.

- Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vector space V.

- For a vector v in V write it as a linear combination of the vectors in the basis:
$$v = \sum_{i=1}^{n} a_i u_i.$$

- The **coordinate vector of** x **relative to** β, denoted $[x]_\beta$ is
$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- Let $T : V \to W$ be linear.

- Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ be basis for V and, respectively, W.

- Then we can write

$$T(v_j) = \sum_{i=1}^{n} a_{ij} w_i \text{ for } 1 \leq j \leq n.$$

- We call the $m \times n$ matrix A defined by the scalars a_{ij} the **matrix representation of T in the ordered bases β and γ**; we write $A = [T]_\beta^\gamma$.

The Matrix Representation of the Sum of two Linear Transformations

Theorem. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U : V \to W$ be linear transformations. Then*

1. $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ *and*

2. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ *for all scalars a.*